Faculty of Business and Economics
Department of Economics

ECON40002
Chris Edmond

Advanced Macroeconomics
Tutorial #2: Solutions

1. **Ramsey-Cass-Koopmans model.** Suppose the planner seeks to maximize the intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t \, u\Big(\frac{C_t}{L}\Big)L, \qquad 0 < \beta < 1$$

subject to the sequence of resource constraints

$$C_t + K_{t+1} = F(K_t, AL) + (1-\delta)K_t, \qquad 0 < \delta < 1$$

given initial $K_0 > 0$. Suppose for simplicity that the labor force L and the level of productivity A are constant. Let c_t, k_t, y_t etc denote consumption, capital, output etc in *per worker* units. Suppose that the period utility function and the production function have their usual properties.

(a) Derive optimality conditions that characterize the solution to the planner's problem. Give intuition for those optimality conditions. Explain how these optimality conditions pin down the dynamics of c_t and k_t.

(b) Now suppose that the production function is Cobb-Douglas, $Y = F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$ with $0 < \alpha < 1$. Derive expressions for the steady state values c^*, k^* and y^* in this economy. How do these steady state values depend on the period utility function? Explain.

(c) What is the steady state savings rate in this economy? Explain how the steady state savings rate depends on the parameters α, β, δ and the level of productivity A. Give intuition for your answers.

(d) Suppose the economy is initially in the steady state you found in (b). Suppose the economy becomes more patient with the discount factor increasing from β to $\beta' > \beta$. Use a phase diagram to explain (i) how this change affects the long-run values of consumption, capital and output, and (ii) how the economy *transitions* to these new long-run values. How would your answers differ if the discount factor fell from β to $\beta' < \beta$?

Solutions:

(a) The planner's problem is to maximize intertemporal utility (per worker)

$$\sum_{t=0}^{\infty} \beta^t \, u(c_t)$$

subject to the sequence of resource constraints

$$c_t + k_{t+1} = F(k_t, A) + (1-\delta)k_t$$

given the initial condition $k_0 > 0$. Notice that I have used constant returns to write the production function as

$$y = \frac{Y}{L} = \frac{F(K, AL)}{L} = F\big(\frac{K}{L}, A\big) = F(k, A)$$

Setting up the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t [F(k_t, A) + (1-\delta)k_t - c_t - k_{t+1}]$$

The key first order conditions for this problem are, for consumption,

$$c_t : \qquad \beta^t u'(c_t) = \lambda_t$$

and for capital,

$$k_{t+1} : \qquad \lambda_t = \lambda_{t+1}\big[F_k(k_{t+1}, A) + 1 - \delta\big]$$

(where F_k denotes the marginal product of capital per worker), and for the multipliers,

$$\lambda_t : \qquad c_t + k_{t+1} = F(k_t, A) + (1-\delta)k_t$$

We also have the transversality condition

$$\lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0$$

Eliminating the multipliers λ_t gives the consumption Euler equation

$$u'(c_t) = \beta u'(c_{t+1})[F_k(k_{t+1}, A) + 1 - \delta]$$

To interpret this condition, let $R_{t+1} \equiv F_k(k_{t+1}, A) + 1 - \delta$ denote the gross return on capital. Then consumption is increasing, $c_{t+1} > c_t$ if and only if $u'(c_{t+1}) < u'(c_t)$ [since $u'(c)$ is strictly decreasing] which happens if and only if

$$1 < \beta R_{t+1}$$

In short, consumption in $t+1$ will be high relative to consumption in t when the return on capital is high relative to time discounting. When the return on capital is relatively high, it makes sense to defer consumption today and invest in physical capital to secure more consumption tomorrow so that $c_{t+1} > c_t$. Rewriting the discount factor $\beta = 1/(1+\rho)$ in terms of the discount rate $\rho > 0$ we can equivalently say that consumption is increasing if and only if

$$\rho + \delta < F_k(k_{t+1}, A)$$

that is, if and only if the marginal product of capital is relatively high.

The consumption Euler equation and the resource constraint are two nonlinear difference equations in c_t, k_t. To pin down the dynamics of c_t, k_t we also need two boundary conditions. One of these is the given initial condition $k_0 > 0$. The other is the transversality condition given above.

(b) If the production function is Cobb-Douglas $F(K, AL) = K^\alpha (AL)^{1-\alpha}$ then $y = F(k, A) = k^\alpha A^{1-\alpha}$ with marginal product $F_k = \alpha(k/A)^{\alpha-1}$. In a steady state with $c_t = c_{t+1} = c^*$ the consumption Euler equation implies

$$1 = \beta[\alpha(k^*/A)^{\alpha-1} + 1 - \delta]$$

which can be solved to get steady state capital per worker

$$k^* = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} A$$

which is proportional to the level of productivity A. Steady state output per worker is then found from the production function

$$y^* = F(k^*, A) = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} A^\alpha A^{1-\alpha} = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} A$$

which is also proportional to the level of productivity A. Note that the steady state capital/output ratio is independent of productivity

$$\frac{k^*}{y^*} = \frac{\alpha}{\rho+\delta}$$

Using the resource constraint $c^* = y^* - \delta k^*$, the steady state consumption/output ratio is then

$$\frac{c^*}{y^*} = 1 - \delta\frac{k^*}{y^*} = \frac{\rho + (1-\alpha)\delta}{\rho+\delta}$$

also independent of productivity. The level of steady state consumption per worker is then

$$c^* = \left(\frac{\rho + (1-\alpha)\delta}{\rho+\delta}\right)\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} A$$

None of these expressions depend on the period utility function $u(c)$ at all. This is an artefact of time-separable preferences (and the absence of trend growth). In this setup, the utility function matters for the short-run dynamics but not for the long-run (steady state) values.

(c) Let the savings rate be $s_t = S_t/Y_t = I_t/Y_t$. In steady state, $s^* = \delta k^*/y^*$ so using our results from part (b) we have

$$s^* = \delta\frac{k^*}{y^*} = \frac{\alpha\delta}{\rho+\delta}$$

The steady state savings rate s^* is increasing in capital's share α, increasing in the depreciation rate δ, and decreasing in the rate of time preference ρ. Intuitively, when capital contributes more to output then savings is higher (other things equal). When the depreciation rate is higher, then in steady state the savings rate is higher to compensate. When the planner discounts the future more (ρ is higher, i.e., the planner is more impatient), then the savings rate is lower.

(d) First note that since the discount factor is $\beta = 1/(1+\rho)$, an increase in the discount factor is equivalent to a decrease in the discount rate. For example, an increase in the discount factor from $\beta = 0.98$ to $\beta' = 0.99$ is a decrease in the discount rate from $\rho \approx 0.02$ to

$\rho' \approx 0.01$. Then using the expressions in part (b) above, it is clear that a decrease in the discount rate from ρ to $\rho' < \rho$ increases steady state capital from k^* to $k^{*\prime}$, say, and hence increases steady state output from y^* to $y^{*\prime}$ and steady state consumption from c^* to $c^{*\prime}$. Then relative to these new steady state levels the economy 'begins' with initial capital per worker $k_0 = k^* < k^{*\prime}$. On 'impact' the level of consumption immediately jumps *down* to $c(0) < c^*$ on the new stable arm going through the new steady state. Capital does not jump on impact because it is predetermined and hence output does not change on impact either. The level of consumption jumps down because the economy is more patient and is saving more. This increase in saving/investment is what allows the economy to build up a new higher level of capital in the long run. As the economy transitions to its new long run, output rises as more capital is accumulated and this allows consumption to rise too. As time passes, consumption keeps increasing, passing the old steady state c^*, and converging in the long run to the new steady state $c^{*\prime} > c^*$.

If instead there is an increase in the discount rate from ρ to $\rho' > \rho$ then steady state capital decreases to $k^{*\prime} < k^*$ and hence steady state output decreases to $y^{*\prime} < y^*$ and hence steady state consumption decreases to $c^{*\prime} < c*$. Then relative to these new steady state levels the economy 'begins' with initial capital per worker $k_0 = k^* > k^{*\prime}$. On 'impact' the level of consumption immediately jumps *up* to $c(0) > c^*$ on the new stable arm going through the new steady state. Capital does not jump on impact because it is predetermined and hence output does not change on impact either. The level of consumption jumps up because the economy is less patient and is saving less. This decrease in saving/investment leads to a long-run decline in the level of capital. As the economy transitions to its new long run, output falls as the capital stock falls and this means consumption falls too. As time passes, consumption keeps decreasing, passing the old steady state c^*, and converging in the long run to the new steady state $c^{*\prime} < c^*$.

2. **Isoelastic utility.** Consider the utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \qquad \sigma > 0$$

(a) Show that $u(c)$ is strictly increasing and strictly concave.

(b) Show that the relative curvature of the utility function is

$$-\frac{u''(c)c}{u'(c)} = \sigma$$

independent of the level of consumption.

(c) Consider the case $\sigma \to 1$. Show that this corresponds to $u(c) \to \log c$. [*Hint*: what is the antiderivative of c^{-1}?]

Solutions:

(a) Note that

$$u'(c) = \frac{(1-\sigma)c^{1-\sigma-1} - 0}{1-\sigma} = c^{-\sigma} > 0$$

and

$$u''(c) = -\sigma c^{-\sigma-1} < 0$$

Hence $u(c)$ is strictly increasing and strictly concave.

(b) Using these derivatives, the relative curvature of the utility function is

$$-\frac{u''(c)c}{u'(c)} = -\frac{-\sigma c^{-\sigma-1}c}{c^{-\sigma}} = \frac{\sigma c^{-\sigma}}{c^{-\sigma}} = \sigma > 0$$

The elasticity of marginal utility $u'(c)$ with respect to consumption is the constant σ.

(c) From l'Hôpital's rule we have that for any $c > 0$ the limit

$$\begin{aligned}
\lim_{\sigma\to 1} \frac{c^{1-\sigma}-1}{1-\sigma} &= \lim_{\sigma\to 1} \frac{\frac{d}{d\sigma}(c^{1-\sigma})}{\frac{d}{d\sigma}(1-\sigma)} \\
&= \frac{\lim_{\sigma\to 1}\{\exp[(1-\sigma)\log c](-\log c)\}}{-1} \\
&= \frac{\exp(0)(-\log c)}{-1} \\
&= \log c
\end{aligned}$$

Another way to see this result is to observe that in general $u'(c) = c^{-\sigma}$ hence in the special case $\sigma \to 1$ marginal utility is just c^{-1}. The level of utility is then the integral of marginal utility, but this is the *definition* of the natural log function, i.e., $\log x \equiv \int_1^x t^{-1}\,dt$.