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ECON40002
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Advanced Macroeconomics
Tutorial #1: Solutions

1-2. To be done in class.

3. **Solow-Swan model.** Consider a standard Solow-Swan model in continuous time with Cobb-Douglas production function $y = k^\alpha$, constant savings rate s, depreciation rate δ, productivity growth g and employment growth n.

   (a) Derive expressions for the steady state values k^*, y^*, c^* in terms of the model parameters s, δ, g, n and α.

   (b) Use a diagram to explain how an increase in s affects k^*, y^*, c^*. Does this change in s increase or decrease long run output and consumption per worker? Explain.

   (c) Use a diagram to explain how an increase in α affects k^*, y^*, c^*. Does this change in α increase or decrease long run output and consumption per worker? Explain.

   Solutions:

   (a) Steady state capital k^* solves

   $$sf(k^*) = (\delta + g + n)k^*$$

   Hence with $y = f(k) = k^\alpha$ we have the solution (ignoring the trivial $k^* = 0$ case)

   $$k^* = \left(\frac{s}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}$$

   And therefore

   $$y^* = \left(\frac{s}{\delta + g + n}\right)^{\frac{\alpha}{1-\alpha}}$$

   $$c^* = (1-s)\left(\frac{s}{\delta + g + n}\right)^{\frac{\alpha}{1-\alpha}}$$

   (b) An increase in s unambiguously increases k^* and hence increases y^*. Whether an increase in s increases c^* depends on where the level of s is relative to the 'golden rule' level (which with this Cobb-Douglas production function, is equal to α, see Lecture 2 slide 18). If $s < \alpha$ then a marginal increase in s increases c^* but if $s > \alpha$ then a marginal increase in s decreases c^*.

(c) An increase in α increases k^* only if the capital/output ratio

$$\frac{k^*}{y^*} = \frac{s}{\delta + g + n}$$

is greater than 1. If $k^*/y^* > 1$ then a higher α increases k^* and hence increases y^* and c^*. In this situation, the parameters are conducive to capital accumulation (the savings rate is high relative to effective depreciation), so less curvature in the production function (α higher) makes for a larger steady state level of capital per effective worker. If $k^*/y^* < 1$ then a higher α decreases k^* and hence decreases y^* and c^*. In this situation, the parameters are not conducive to capital accumulation (the savings rate is low relative to effective depreciation), so less curvature in the production function makes for a lower steady state level of capital per effective worker.

4. **Linear production function.** Suppose the production function is linear $y = k$ and for simplicity suppose no productivity or employment growth, $g = n = 0$. Does the Solow-Swan model have a steady state capital stock in this setting? Why or why not? Explain the dynamics of $k(t)$ in this economy. How do these dynamics depend on the values of s and δ? Explain. What standard assumptions about the production function does this example violate?

   Solution:

   With $y = f(k) = k$, capital accumulation is given by

$$\dot{k}(t) = sk(t) - \delta k(t) = (s - \delta)k(t)$$

   (since $g = n = 0$). Steady states k^* are given by points such that $\dot{k}(t) = 0$. In this case there is generally no steady state except the trivial one at $k^* = 0$. In the 'knife-edge' case where $s = \delta$, any k is a steady state. More generally, for $s \neq \delta$, either (i) the savings rate is greater than depreciation so that $\dot{k}(t) = (s - \delta)k(t) > 0$ for all t and the capital stock grows without bound, or (ii) the savings rate is less than depreciation so that $\dot{k}(t) = (s - \delta)k(t) < 0$ for all t and the capital stock shrinks towards 0. This linear production $y = f(k) = k$ function has positive marginal product, $f'(k) = 1$, but does not exhibit diminishing returns $f''(k) = 0$. Moreover, since $f'(k) = 1$ for all k, this linear production function does not satisfy the usual Inada conditions [$f'(0) = \infty$ and $f'(\infty) = 0$]. In particular, the failure of the first Inada condition $f'(0) = \infty$ exposes the economy to $k(t) \to 0$ when $s < \delta$ while the failure of the second Inada condition $f'(\infty) = 0$ allows the economy to experience unbounded growth $k(t) \to \infty$ when $s > \delta$. In this latter case, the basic conclusion of the Solow model — i.e., that capital accumulation alone cannot sustain long run growth — is overturned.

5. **Inada conditions.** Consider a production function in intensive form $y = f(k)$. Briefly explain the role played by the Inada conditions $f'(0) = \infty$ and $f'(\infty) = 0$ in analyzing the Solow-Swan model. In particular, suppose $f'(k) > 0$ and $f''(k) < 0$ but that the Inada conditions are *not* satisfied. What possibilities does this lead to?

   Solution:

   As alluded to above, the Inada conditions guarantee the existence of an interior steady state $k^* > 0$. For simplicity, again suppose that $g = n = 0$. Then if

$$sf'(0) < \delta$$

we have that investment is never more than depreciation so that the only steady state is the trivial $k^* = 0$ and capital shrinks toward this steady state $k(t) \to 0$. At the other extreme, if

$$sf'(\infty) > \delta$$

we have that investment is never less than depreciation, again the only steady state is the trivial $k^* = 0$ but capital grows without bound, $k(t) \to \infty$, away from this steady state. Notice that the usual Inada conditions $f'(0) = \infty$ and $f'(\infty) = 0$ are *sufficient* to ensure these cases do not arise, but are not *necessary*. The necessary conditions are that

$$f'(\infty) < \frac{\delta}{s} < f'(0)$$

The linear production function $y = f(k) = k$ implies that *one* of these Inada conditions fails, which one depends on the magnitudes of s and δ.