ECON40002
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Advanced Macroeconomics
Tutorial #11: Solutions

1. **Optimal insurance and deposit contracts.** Consider the Diamond-Dybvig model. There are three dates $\{0, 1, 2\}$ and a unit mass of ex ante identical investors and a single bank. Each of the investors has an endowment of 1 to invest at date $t = 0$. The type of each investor is revealed at date $t = 1$. A fraction $\alpha \in (0, 1)$ are *impatient* and consume only at $t = 1$. The remaining fraction are *patient* and indifferent between consuming at either $t = 1$ or $t = 2$. An individual's realised type is her own private information.

   Funds invested for *two* periods earn a gross return $R > 1$ (an *illiquid project*). Funds invested for only one period earn a gross return of 1 (i.e., the investor just gets their funds back).

   Each investor has the CRRA utility function

   $$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \qquad \sigma > 1$$

   (a) Set up the optimization problem the solution of which gives the efficient amount of risk-sharing (optimal insurance) between impatient and patient investors.

   (b) Suppose the following parameter values: $\alpha = 0.5$, $R = 4$ and $\sigma = 2$. Using these parameter values, solve the optimization problem for the payments c_1^*, c_2^* to impatient and patient investors.

   (c) Explain how the optimal insurance scheme can be implemented by a liquid deposit contract with the bank that pays returns r_1, r_2 on dates $t = 1$ and $t = 2$ respectively. What would the values of r_1, r_2 have to be?

   (d) Calculate the ex ante expected utility to an investor who enters into this deposit contract. Is this higher or lower than the ex ante expected utility of an investor who just invests and holds the illiquid asset? Explain. How would your answer change (if at all) if the investors were risk neutral (e.g., $u(c) = c$)? Explain.

   (e) Explain the *sequential service constraint* facing the bank if it offers deposit contracts. Explain why the bank is prone to a run. If the return on the deposit contract paid in the first period r_1 is the value calculated in part (c), what is the maximum number of withdrawals f^* beyond which any individual patient investor will find it optimal to withdraw? [the '*tipping point*']

   SOLUTIONS.

   (a) The optimization problem is to choose c_1, c_2 both nonnegative to maximize

   $$\alpha u(c_1) + (1 - \alpha) u(c_2)$$

subject to the resource constraint

$$\alpha c_1 + (1-\alpha)\frac{c_2}{R} \leq 1$$

and the incentive constraint

$$u(c_1) \leq u(c_2)$$

(b) Guessing that the incentive constraint is slack, the first order condition for the problem is

$$u'(c_1) = u'(c_2)R$$

Since marginal utility is $u'(c) = c^{-\sigma}$ we can write the first order condition as

$$c_1^{-\sigma} = c_2^{-\sigma}R \qquad \Leftrightarrow \qquad c_2 = R^{\frac{1}{\sigma}}c_1 > c_1$$

since $R > 1$ and $\sigma > 0$. Therefore $u(c_2) > u(c_1)$ and the incentive constraint is indeed slack. Combining this with the resource constraint gives

$$\alpha c_1 + (1-\alpha)R^{\frac{1}{\sigma}-1}c_1 = 1$$

and solving for c_1 gives

$$c_1^* = \frac{1}{\alpha + (1-\alpha)R^{\frac{1-\sigma}{\sigma}}}$$

and therefore

$$c_2^* = \frac{R^{\frac{1}{\sigma}}}{\alpha + (1-\alpha)R^{\frac{1-\sigma}{\sigma}}}$$

Plugging in $\alpha = 0.5$, $R = 4$ and $\sigma = 2$ then gives

$$c_1^* = \frac{1}{0.5 + (1-0.5)4^{-1/2}} = 1.33 > 1$$

and

$$c_2^* = \frac{4^{1/2}}{0.5 + (1-0.5)4^{-1/2}} = 2.67 < R = 4$$

(c) For the deposit contract, we take 1 from all investors and pay r_1 to early withdrawals and r_2 to investors who keep leave their deposits in place for two periods. In the meantime, the bank takes the deposits and uses them for the project that delivers R per unit but only if funds are in place for two periods. If the fraction of early withdrawals f just equals the fraction of impatient types $f = \alpha$ then we can implement the optimal insurance arrangement by setting $r_1 = c_1^*$ and $r_2 = c_2^*$ as given in part (b) above. That is, $r_1 = 1.33 > 1$ and $r_2 = 2.67 < R = 4$.

(d) An investor in autarky (who invests and holds the illiquid project but who has to pull funds out at date 1 if they're unlucky and turn out to be impatient) has consumption $c_1 = 1$ and $c_2 = R = 4$ and so their expected utility is

$$EU_{autarky} = 0.5u(1) + 0.5u(R) = 0.3750$$

using $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ with $\sigma = 2$. But an investor who has the deposit contract (with $f = \alpha$) has expected utility

$$EU_{deposit} = 0.5u(1.33) + 0.5u(1.67) = 0.4375$$

and so they prefer the deposit arrangement, at least if only the patient types withdraw early. By contrast, a risk neutral investor with $u(c) = c$ would have

$$EU_{autarky} = 0.5 \times 1 + 0.5 \times 4 = 2.5$$

while for the deposit contract

$$EU_{deposit} = 0.5 \times 1.33 + 0.5 \times 1.67 = 2$$

and so the risk neutral investor prefers autarky (the reduction in return for being patient is too big, it is after all partly to provide insurance to risk averse people and the risk neutral investor doesn't value that). Don't make the mistake of comparing the level of expected utility of the risk averse investor to that of the risk neutral investor (e.g., 0.3750 to 2.5). We can take arbitrary positive monotone increasing transformations of the underlying utility function $u(c)$ (e.g., adding positive constants, multiplying by positive numbers etc) without affecting an individual's rank ordering of outcomes, so interpersonal comparisons of utility levels are not informative.

(e) The sequential service constraint requires that individuals trying to withdraw get paid out depending only on their place in the queue for deposits (so a patient type who arrives before an impatient type gets paid first even though the impatient type has greater need). If the fraction who withdraw early is $f \geq \alpha$ (at least all impatient types withdraw early), then the sequential service constraint can be written

$$r_2(f) = \max\left[0,\ R\frac{1 - fr_1}{1 - f}\right]$$

After early withdrawals there is $1 - fr_1$ remaining in the deposit accounts (or nothing if f is too high). Supposing there's anything left, these funds earn $R(1 - fr_1)$ in total after the second period and this has to be divided amongst the remaining $1 - f$ investors. What level of f is too high (the tipping point)? Well, investors withdraw if $r_2(f) \leq r_1$ or equivalently, after rearranging the equation above, if

$$f \geq f^* \equiv \frac{1}{r_1}\left(\frac{R - r_1}{R - 1}\right)$$

Plugging in the values we have

$$f^* = \frac{1}{1.33}\left(\frac{4 - 1.33}{4 - 1}\right) = 0.67$$

Any $f > 0.67$ makes it optimal for an investor to withdraw. Note: that the fraction of individuals that withdraw in a *pure-strategy Nash equilibrium* is either $f = 0.5$ (only impatient types withdraw) or $f = 1$ (all withdraw).

2. **Structured finance.** Suppose there are two bonds and that each pays \$1 cash or not. The probability of getting \$1 is 0.8 and is independent across bonds.

(a) Explain how a financial intermediary can sell prioritized junior j and senior s claims to \$1 against the possible cash flows from a portfolio of these two bonds. In your answer, give the possible realizations of the cash flows, the probabilities of these events, and the payments made to junior and senior claims in each event. How much would a risk neutral investor be prepared to pay for the j and s claims. Is this more or less than they would pay for the underlying bonds? Explain.

(b) Now suppose there are *two pools each of two bonds* each as in part (a) above. Each pool has junior and senior claims. Explain how a financial intermediary can sell prioritized junior j_j and senior s_j claims to \$1 against the possible cash flows from a portfolio formed from the junior tranches j_1 and j_2 from each pool. What pattern of cash flows leads to senior claim in the second round of securitization being paid or not paid? Give the possible realizations of the cash flows, the probabilities of these events, and the payments made to the j_j and s_j claims from the second round of securitization. Would a risk neutral investor pay more for a senior claim in the first round of securitization (s_1 or s_2) or for a senior claim in the second round (s_j)? Explain.

(c) Now suppose there are two bonds as in part (a) except that the underlying bonds payments are perfectly positively correlated. Give the possible realizations of the cash flows, the probabilities of these events, and the payments made to junior and senior claims in each event. Would a risk neutral investor be prepared to pay a premium for senior claims? Explain. What if the underlying bond payments are instead perfectly *negatively* correlated, would your answers change? Would a risk averse investor view things differently?

SOLUTIONS:

(a) The possible realizations and their probabilities are given in the table below. The calculations of the probabilities of each state use the fact that the probability of getting 1 is independent across bonds. In any state of the world where either bond pays out, we pay 1 to the senior claim. Only if both bonds pay out do we pay 1 to the junior claim. In this sense, the junior claim is the *residual claimant* to the cash flow from the package of bonds (like *equity*).

realization	$\{0,0\}$	$\{0,1\}$	$\{1,0\}$	$\{1,1\}$
probability	.04	.16	.16	.64
payment $\{j,s\}$	$\{0,0\}$	$\{0,1\}$	$\{0,1\}$	$\{1,1\}$

The probability of the junior claim being paid is therefore $\Pr(j=1) = .64$ while the probability of the senior claim being paid is $\Pr(s=1) = .64 + .16 + .16 = .96$ (i.e., $= 1 - .04$). A risk neutral investor would be willing to pay at most .64 for the junior claim and at most .96 for the senior claim. Therefore they would be willing to pay more for the senior claim than for one of the underlying bonds (due to the protection offered by the junior claim) but the junior claim is worth less than one of the underlying bonds.

(b) Now we take the payments against the junior claims j_i for $i = 1, 2$ pools each of 2 bonds as in part (a). Below are the realizations, probabilities and cash flows in the second round of securitization.

realization $\{j_1, j_2\}$	$\{0,0\}$	$\{0,1\}$	$\{1,0\}$	$\{1,1\}$
probability	.1296	.2304	.2304	.4096
payment $\{j_j, s_j\}$	$\{0,0\}$	$\{0,1\}$	$\{0,1\}$	$\{1,1\}$

The probability of the junior claim in the second round being paid is therefore $\Pr(j_j = 1) = .4096$ while the probability of the senior claim in the second round being paid is

$\Pr(s_j = 1) = 1 - .1296 = .8704$. In order for the junior claim in the first round to be paid out, there has to be no default in the pool on which that claim is written. So in order for the senior claim in the second round to be paid out, there has to be *no default in at least one* of the two pools of bonds. A risk neutral investor would pay at most .8704 for a senior claim in the second round, i.e., less than the .96 they'd be willing to pay for a senior claim in the first round. Although safer than the underlying bonds and the junior claims from the first round, the senior claim in the second round is still riskier than the senior claims in the first round.

(c) If the underlying bonds are perfectly *positively* correlated, then either both bonds pay out (with probability .80) or neither does (with probability .20). In this case there is no possibility of using prioritization (i.e., a *capital structure*) to protect a senior claim. Since there is no possibility of using prioritization, a risk neutral investor would pay at most .80 for a claim, the same as for the underlying bonds. If the bonds are instead perfectly *negatively* correlated, then a pool of two such bonds pays out 1 with probability 1.00 (since if one doesn't pay, the other does). Thus a claim to a pool of these two bonds can deliver 1 for sure and a risk neutral investor would be willing to pay 1 for such a claim (more than .80). Notice therefore that it is not correlation *per se* across the underlying bonds that destroys the ability to protect a senior claim, it is more specifically positive correlation that is the problem. Negative correlation across the underlying bonds makes it easier not harder to protect the senior claim (as always, at the cost of making the junior claim worth less). In general, a risk averse investor will always need to be compensated for risk by being able to buy a security at a price lower than the risk neutral investor would be prepared to pay. How much of a discount depends on the curvature in their utility function. For CRRA utility with coefficient σ, the required discount is proportional to $\sigma/2$ times the variance of the cash flow [at least for small risks].