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ECON40002
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Advanced Macroeconomics

Tutorial #10: Solutions

1. **Practice with Bellman values.** Consider a discrete time setting $t = 0, 1, 2, \dots$ where a risk-neutral worker receives period utility w_t from a stream of wage payments. Let W_t denote the present value of these payments discounted at constant rate $r > 0$

$$W_t \equiv \sum_{s=t}^{\infty} e^{-r(s-t)}\, w_s$$

(supposing the sum is well-defined, e.g., that $w_t \to \bar{w} > 0$ as $t \to \infty$).

(a) Show that the present value W_t satisfies the difference equation

$$W_t = w_t + e^{-r}\, W_{t+1} \tag{1}$$

(b) Now let the period length be $\Delta > 0$ so that $t = 0, \Delta, 2\Delta, \dots$ with total wage payment $w_t\Delta$ over a period of length Δ. Explain why the present value W_t now satisfies the difference equation

$$W_t = w_t\Delta + e^{-r\Delta}\, W_{t+\Delta}$$

(c) Now let the period length $\Delta \to 0$. Show that in this continuous time limit, the present value satisfies the differential equation

$$rW(t) = w(t) + \dot{W}(t) \tag{2}$$

Now consider again the simple discrete time $t = 0, 1, 2, \dots$ setting with wage payments w_t. Suppose that the wage is stochastic and can take on two values, $w_t \in \{w_l, w_h\}$ with $0 < w_l < w_h$. Suppose that if the current wage is w_l then with probability p_{ll} the wage next period remains w_l but with probability $1 - p_{ll}$ the wage next period jumps up to w_h. Likewise if the current wage is w_h then with probability p_{hh} the wage next period remains w_h but with with probability $1 - p_{hh}$ the wage next period drops to w_l. In short

$$p_{ll} = \text{Prob}[w_{t+1} = w_l \,|\, w_t = w_l]$$
$$p_{hh} = \text{Prob}[w_{t+1} = w_h \,|\, w_t = w_h]$$

(d) Let W_t^l denote the present value of wages conditional on $w_t = w_l$. Likewise let W_t^h denote the present value of wages conditional on $w_t = w_h$. What system of difference equations do these values solve?

(e) Now again consider periods of length $\Delta > 0$ and suppose that over a period of length Δ the probability of remaining in state l is $p_{ll} = e^{-\lambda_h \Delta}$ for some $\lambda_h > 0$ and similarly the probability of remaining in state h is $p_{hh} = e^{-\lambda_l \Delta}$ for $\lambda_l > 0$. Again let the period length $\Delta \to 0$. What system of differential equations do $W^l(t), W^h(t)$ solve? How do the present values depend on λ_l, λ_h in steady state? Explain.

SOLUTIONS:

(a) First write the sum in terms of the date t contribution and all remaining contributions

$$W_t = w_t + \sum_{s=t+1}^{\infty} e^{-r(s-t)}\, w_s$$

Then factor out a e^{-r} term from the sum on the RHS as follows

$$W_t = w_t + e^{-r} \sum_{s=t+1}^{\infty} e^{r} e^{-r(s-t)}\, w_s$$

Then tidying up the exponents inside the sum

$$W_t = w_t + e^{-r} \sum_{s=t+1}^{\infty} e^{-r(s-(t+1))}\, w_s$$

But then note that

$$\sum_{s=t+1}^{\infty} e^{-r(s-(t+1))}\, w_s = W_{t+1}$$

So we have, as required,

$$W_t = w_t + e^{-r}\, W_{t+1}$$

(b) Imagine that in our original discrete time setting the 'length of a period' is 'one year'. Then w_t refers to the wage earned over a period of one year and the discount factor $0 < e^{-r} < 1$ means that a unit of wages today is worth e^{-r} wages in one year's time. If we set $\Delta = 1/4$ so that we re-parameterize the length of a period to 'one quarter', then the equivalent *quarterly* wage is $w_t\Delta = w_t/4$ and the discount factor over one quarter is $e^{-r\Delta} = e^{-r/4}$. Of course if we sum up the four quarterly wages $w_t/4 + w_t/4 + w_t/4 + w_t/4$ we get back our original annual wage w_t and if we cumulate our discount factor $e^{-r/4}e^{-r/4}e^{-r/4}e^{-r/4}$ we get back our original annual discount factor e^{-r}. Likewise by choosing $\Delta = 1/365$ we effectively get a 'daily' model with daily wage $w_t/365$ and daily discount factor $e^{-r/365}$. In any case, given this flow wage $w_t\Delta$ and discount factor $e^{-r/\Delta}$, by following the same steps as in part (a) but replacing t and $t+1$ with t and $t+\Delta$ we get the required

$$W_t = w_t\Delta + e^{-r\Delta}\, W_{t+\Delta}$$

(c) First form the finite difference

$$W_{t+\Delta} - W_t = W_{t+\Delta} - w_t\Delta - e^{-r\Delta}\, W_{t+\Delta}$$

Then collect terms and divide both sides by $\Delta > 0$

$$\frac{W_{t+\Delta} - W_t}{\Delta} = -w_t + \frac{1 - e^{-r\Delta}}{\Delta} W_{t+\Delta}$$

Taking the limit as $\Delta \to 0$, the LHS becomes $\dot{W}(t)$ while on the RHS we use l'Hôpital's rule

$$\lim_{\Delta\to 0} \frac{1 - e^{-r\Delta}}{\Delta} = \lim_{\Delta\to 0} \frac{re^{-r\Delta}\log e}{1} = +r$$

so that we get in the limit

$$\dot{W}(t) = -w(t) + rW(t)$$

which is the same as

$$rW(t) = w(t) + \dot{W}(t)$$

(d) Let $W_t(x)$ denote the present value of wages conditional on the current wage being $x \in \{w_l, w_h\}$, that is

$$W_t(x) \equiv \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} e^{-r(s-t)} w_s \,\middle|\, w_t = x \right\}$$

In this notation, $W_t^l = W_t(w_l)$ and $W_t^h = W_t(w_h)$. Then following the same steps as in part (a) we have

$$W_t(x) = x + e^{-r}\mathbb{E}_t \left\{ W_{t+1}(x') \mid x \right\}$$

Now if $x = w_l$ then $x' = w_l$ with probability p_{ll} but $x' = w_h$ with probability $1 - p_{ll}$ so

$$W_t^l = w_l + e^{-r} \left\{ p_{ll} W_{t+1}^l + (1 - p_{ll}) W_{t+1}^h \right\}$$

And if $x = w_h$ then $x' = w_h$ with probability p_{hh} but $x' = w_l$ with probability $1 - p_{hh}$ so

$$W_t^h = w_h + e^{-r} \left\{ (1 - p_{hh}) W_{t+1}^l + p_{hh} W_{t+1}^h \right\}$$

These last two expressions are a system of two linear difference equations in W_t^l, W_t^h.

(e) Over a period of length $\Delta > 0$ and using the given parameterizations — i.e., that $p_{ll} = e^{-\lambda_h \Delta}$ etc — we have

$$W_t^l = w_l \Delta + e^{-r\Delta} \left\{ e^{-\lambda_h \Delta} W_{t+\Delta}^l + (1 - e^{-\lambda_h \Delta}) W_{t+\Delta}^h \right\}$$

Now form the finite difference

$$W_{t+\Delta}^l - W_t^l = W_{t+\Delta}^l - w_l \Delta - e^{-(r+\lambda_h)\Delta} W_{t+\Delta}^l - e^{-r\Delta}(1 - e^{-\lambda_h \Delta}) W_{t+\Delta}^h$$

Divide both sides by $\Delta > 0$ and collect terms

$$\frac{W_{t+\Delta}^l - W_t^l}{\Delta} = -w_l + \frac{1 - e^{-(r+\lambda_h)\Delta}}{\Delta} W_{t+\Delta}^l - \frac{e^{-r\Delta}(1 - e^{-\lambda_h \Delta})}{\Delta} W_{t+\Delta}^h$$

Using l'Hôpital's rule we have that as $\Delta \to 0$

$$\frac{1 - e^{-(r+\lambda_h)\Delta}}{\Delta} \to +(r + \lambda_h)$$

and

$$\frac{e^{-r\Delta}(1 - e^{-\lambda_h \Delta})}{\Delta} \to +\lambda_h$$

Hence in the limit we get

$$\dot{W}^l(t) = -w_l + (r + \lambda_h) W^l(t) - \lambda_h W^h(t)$$

or

$$rW^l(t) = w_l + \dot{W}^l(t) + \lambda_h (W^h(t) - W^l(t))$$

Following the same steps as above for the difference equation in W_t^h gives

$$rW^h(t) = w_h + \dot{W}^h(t) + \lambda_l(W^l(t) - W^h(t))$$

Notice that in this parameterization $\lambda_h > 0$ is the flow probability of switching from l to h and λ_l is likewise the flow probability of switching from h to l. The term 'flow probability' is used to emphasize that these parameters are not 'proper probabilities' of an event. While $p_{ll} = e^{-\lambda_h \Delta}$ is the probability of remaining in state l over an interval of length $\Delta > 0$ and is a proper probability between 0 and 1, the parameter λ_h is really the *elasticity* of this probability as $\Delta \to 0$ and can be more than one.

In steady state we have $\dot{W}^l(t) = \dot{W}^h(t) = 0$ so that

$$rW^l = w_l + \lambda_h(W^h - W^l)$$

$$rW^h = w_h + \lambda_l(W^l - W^h)$$

Solving these two equations in two unknowns gives the steady state values

$$rW^l = \frac{r + \lambda_l}{r + \lambda_l + \lambda_h} w_l + \frac{\lambda_h}{r + \lambda_l + \lambda_h} w_h$$

and

$$rW^h = \frac{\lambda_l}{r + \lambda_l + \lambda_h} w_l + \frac{r + \lambda_h}{r + \lambda_l + \lambda_h} w_h$$

So rW^l and rW^h are simply weighted averages of w^l, w^h with the weights depending on the *relative probability* of being in each state as determined by λ_l/λ_h. Notice that, because of discounting, W^l gives more weight to the current realization w^l and W^h gives more weight to the current realization w^h. Put differently, even if $\lambda_l/\lambda_h = 1$ we don't have equal weights on w^l, w^h unless $r \to 0$.

2. **Mortensen-Pissarides model.** Consider a search model of the labor market in continuous time $t \geq 0$. Risk neutral workers and firms discount at constant rate $r > 0$. Workers and firms are matched via a standard constant-returns-to-scale matching function $F(u, v)$ where $u(t)$ denotes the unemployment rate and $v(t)$ the vacancy rate at time t. When a match forms, a firm is able to produce a constant amount of output $z > 0$. The worker receives a wage of $w(t)$ and the firm makes a flow profit of $z - w(t)$. Job matches between workers and firms are destroyed at an exogenous rate $\delta > 0$. Firms can create jobs by posting vacancies with a flow cost κz proportional to z. There is free-entry into job creation. When unemployed, workers receive constant flow utility $b \leq w(t)$ from unemployment benefits.

   (a) Let $V(t), J(t)$ denote the value to a firm of a vacancy and a filled job respectively and let $U(t), W(t)$ denote the value to a worker of unemployment and employment respectively. What are the Bellman equations that describe these four values? Provide an intuitive explanation for each of these equations.

   (b) Explain how the Bellman equations for $U(t), W(t)$ relate to those you derived in Question 1 part (e). Give as much detail as you can.

Now suppose that wages are determined by Nash-Bargaining between a worker and firm such that in equilibrium the worker's surplus is a constant fraction $\beta \in (0,1)$ of the total match surplus

$$W(t) - U(t) = \beta S(t), \qquad S(t) \equiv W(t) - U(t) + J(t) - V(t)$$

Suppose also that free entry drives the value of a vacancy to $V(t) = 0$ for all t.

(c) Let the matching function be $F(u,v) = u^{\alpha}v^{1-\alpha}$. Explain how the steady state wage, labor market tightness $\theta = v/u$ and unemployment rate are determined.

(d) Now suppose the parameter values $r = 0.01$, $z = \kappa = 1$, $b = 0.4$ and $\alpha = \beta = 0.5$. Calculate the steady state wage, labor market tightness, unemployment rate, vacancy rate, and vacancy filling rate $q = F(u,v)/v$. If productivity increases from $z = 1$ to $z = 1.1$ what happens to each of these variables? What about if r increases from $r = 0.01$ to $r = 0.02$? Give intuition for your answers.

(e) Now consider what happens if the wage is fixed at some exogenous level $\bar{w}$. Suppose in particular that $\bar{w}$ equals the value you found in part (d) for $z = 1$ and then productivity increases to $z = 1.1$ holding the wage fixed at $\bar{w}$. What happens to labor market tightness and the unemployment and vacancy rates? How if at all do your answers differ to those you found in part (d)? Explain.

SOLUTIONS:

(a) The Bellman equations for a firm are

$$rJ(t) = z - w(t) + \dot{J}(t) + \delta(V(t) - J(t))$$

and

$$rV(t) = -\kappa z + \dot{V}(t) + q(\theta(t))(J(t) - V(t))$$

The current value of having a filled job $rJ(t)$ is given by the flow profit $z - w(t)$ plus 'capital gain' $\dot{J}(t)$ plus with exogenous flow probability δ the job is destroyed and the firm switches from $J(t)$ to $V(t)$ (which in equilibrium will entail a payoff loss). Likewise the current value of having a vacancy $rV(t)$ is given by the flow cost of keeping a vacancy open $-\kappa z$ plus capital gain $\dot{V}(t)$ plus with endogenous flow probability $q(\theta(t))$ there is a match and the vacancy is filled so that the firm switches from $V(t)$ to $J(t)$ (which in equilibrium will entail a payoff gain). The corresponding Bellman equations for a worker are

$$rW(t) = w(t) + \dot{W}(t) + \delta(U(t) - W(t))$$

and

$$rU(t) = b + \dot{U}(t) + f(\theta(t))(W(t) - U(t))$$

The current value of having a job $rW(t)$ is given by the flow wage $w(t)$ plus capital gain $\dot{W}(t)$ plus with exogenous flow probability δ the job is destroyed and the worker switches from $W(t)$ to $U(t)$ (which in equilibrium will entail a loss). Likewise the current value of being unemployed $rU(t)$ is given by the flow unemployment benefit b plus capital gain $\dot{U}(t)$ plus with endogenous flow probability $f(\theta(t))$ there is a match and the worker switches from $U(t)$ to $W(t)$ (which in equilibrium will entail a gain).

(b) If we let $w_l = b$ and $w_h = w(t)$ and let $U(t) = W^l(t)$ and $W(t) = W^h(t)$ then these are the same Bellman equations as in Question 1 part (e) above. The flow probability of leaving employment is δ, corresponding to λ_l above. The flow probability of leaving unemployment is $f(\theta(t))$, corresponding to λ_h above. Although the wage $w(t)$ and job finding rate $f(\theta(t))$ are endogenous here, and potentially time-varying, that doesn't affect the interpretation of the Bellman equations.

(c) In steady state we have the Bellman equations for the firm

$$rJ = z - w + \delta(V - J), \qquad \text{and} \qquad rV = -\kappa z + q(\theta)(J - V)$$

and for the worker

$$rW = w + \delta(U - W), \qquad \text{and} \qquad rU = b + f(\theta)(W - U)$$

And from Nash-Bargaining

$$W - U = \frac{\beta}{1-\beta}(J - V)$$

The firm's Bellman equation for a filled job gives

$$J = \frac{z - w + \delta V}{r + \delta}$$

and since $V = 0$ from free-entry, we also have

$$J = \frac{z - w}{r + \delta} = \frac{\kappa z}{q(\theta)}$$

Rearranging this

$$\boxed{w = z - (r + \delta)\frac{\kappa z}{q(\theta)}}$$

The wage is the marginal productivity of the worker z less the costs of hiring through the frictional labor market. This 'marginal productivity condition' is a downward sloping relationship between θ and w (since $q(\theta)$ is decreasing in θ). With the given matching function $F(u, v) = u^\alpha v^{1-\alpha}$ we have $q = F(u, v)/v$ so that $q(\theta) = \theta^{-\alpha}$ and $f = F(u, v)/u$ so that $f(\theta) = \theta^{1-\alpha}$. Given this, we can write the marginal productivity condition as

$$w = z - (r + \delta)\kappa z \theta^\alpha$$

Turning now to the worker side of things, from Nash-Bargaining

$$W - U = \frac{\beta}{1-\beta}(J - V) = \frac{\beta}{1-\beta}\frac{\kappa z}{q(\theta)}$$

Hence value of unemployment is

$$rU = b + f(\theta)\frac{\beta}{1-\beta}\frac{\kappa z}{q(\theta)} = b + \frac{\beta}{1-\beta}\kappa z \theta$$

since $f(\theta) = \theta q(\theta)$. Then from the worker's Bellman equation

$$W = \frac{w + \delta U}{r + \delta}$$

so that

$$W - U = \frac{w - rU}{r + \delta}$$

Using Nash-Bargaining again

$$\frac{w - rU}{r + \delta} = \frac{\beta}{1-\beta} J = \frac{\beta}{1-\beta} \frac{z - w}{r + \delta}$$

Hence

$$w - rU = \frac{\beta}{1-\beta}(z - w)$$

So that on using the expression for rU above

$$w = \beta z + (1-\beta) rU = \beta z + (1-\beta)\left[b + \frac{\beta}{1-\beta}\kappa z \theta\right]$$

Which simplifies to

$$\boxed{w = (1-\beta)b + \beta(1 + \kappa\theta)z}$$

This is the 'wage curve,' an upward sloping relationship between θ and w. Together, the wage curve and the marginal productivity condition are two equations that we can solve for w, θ in terms of the parameters. Given labor market tightness determined in this way, we can then back out the unemployment rate u from the Beveridge curve

$$u = \frac{\delta}{\delta + f(\theta)} = \frac{\delta}{\delta + \theta^{1-\alpha}}$$

and back out vacancies v from $v = \theta u$.

(d) The attached Matlab file `tutorial10.m` implements this solution for the given parameter values. The steady state wage and labor market tightness are $w = 0.964$ (a bit less than marginal productivity $z = 1$) and $\theta = 0.527$. We then have job finding rate $f(\theta) = \theta^{1-\alpha} = 0.726$ and so from the Beveridge curve unemployment $u = \delta/(\delta + f(\theta)) = 0.052$ or 5.2%. The vacancy rate is then $v = \theta u = 0.0275$ and the vacancy filling rate is $q(\theta) = \theta^{-\alpha} = 1.377$ (recall that this is the 'flow probability' — really the *elasticity* with respect to time of the probability of changing from V to J — so values greater than 1 are permitted).

With higher productivity, $z = 1.1$ the wage rises to $w = 1.059$ and labor market tightness increases to $\theta = 0.561$. This is because although the wage curve shifts up and the marginal productivity condition shifts out (with offsetting implications for labor market tightness), the effect on the marginal productivity condition is larger so that on net θ rises. Both effects drive wages higher. With θ higher the job finding rate is also higher $f(\theta) = \theta^{1-\alpha} = 0.749$ and so steady state unemployment is lower $u = \delta/(\delta + f(\theta)) = 0.051$ as we rotate counter-clockwise along the Beveridge curve. Vacancies are also higher $v = \theta u = 0.0285$ and the vacancy filling rate is correspondingly lower $q(\theta) = 1.335$.

With a higher discount rate $r = 0.02$, the wage is lower at $w = 0.957$ as is labor market tightness at $\theta = 0.514$. This is because, by making job creation more expensive (in discounted terms), the increase in r shifts the marginal productivity condition down along an unchanged wage curve thereby reducing both w and θ. With θ lower the job finding rate is also lower $f(\theta) = \theta^{1-\alpha} = 0.717$ and so steady state unemployment is higher $u = \delta/(\delta + f(\theta)) = 0.053$ as we rotate clockwise along the Beveridge curve. Vacancies are also lower $v = \theta u = 0.0272$ and the vacancy filling rate is correspondingly higher $q(\theta) = 1.395$.

(e) Suppose we fix the wage at some constant level $\bar{w}$. Then we no longer have Nash-Bargaining and hence no longer have a wage curve. Instead we simply have the marginal productivity condition

$$\bar{w} = z - (r + \delta)\frac{\kappa z}{q(\theta)}$$

which we can solve for θ. If we fix $\bar{w} = 0.964$ as in part (d) above we get the same steady state $\theta = 0.527$ and hence same unemployment rate $u = 0.052$ etc. But now if z increases to $z = 1.1$ we get a much larger increase in labor market tightness, to $\theta = 6.14$, and hence unemployment falls by a larger amount, to $u = 0.016$. This is because with the wage fixed at $\bar{w}$ there is no upward sloping wage curve to mitigate the effects of higher productivity on labor market tightness (i.e., there is no feedback from z to w to θ). Since firms keep the extra profits from the higher productivity, the overall incentives for job creation are much stronger. By contrast, with Nash-Bargaining the wage would rise, so that workers share in some of the gains associated with higher z.

Similarly, if we fix $\bar{w} = 0.964$ as in part (d) but now increase the discount rate to $r = 0.02$ we again get bigger movements. Now labor market tightness falls to $\theta = 0.366$ with unemployment rising to $u = 0.062$. Again the intuition is that without the wage curve, there is no change in w to mitigate the effects of the adverse change in job creation conditions. With the rigid wage, the firm bears most of the brunt of the increase in r but as a result is less inclined to create jobs and so steady state unemployment is higher. By contrast, with Nash-Bargaining the wage would fall so that workers share in some of the losses associated with higher r.