Advanced Macroeconomics
Problem Set #3: Solutions

1. **Inflation dynamics under optimal monetary policy.** Suppose the monetary authority cannot commit to future actions and seeks to minimize the one-period loss function

$$L = \frac{1}{2}\left(\hat{x}_t^2 + \lambda \hat{\pi}_t^2\right), \qquad \lambda > 0$$

subject to the new Keynesian Phillips curve

$$\hat{\pi}_t = \beta \mathbb{E}_t\left\{\hat{\pi}_{t+1}\right\} + \kappa \hat{x}_t + u_t$$

where the cost push shocks u_t follow a stationary AR(1) process

$$u_{t+1} = \phi u_t + \epsilon_{t+1}, \qquad 0 \leq \phi < 1$$

and where the innovations ϵ_t are IID normal with mean zero and variance σ_u^2.

(a) Derive the monetary authority's optimal discretionary policy for inflation and the output gap and the dynamics of inflation implied by this discretionary policy.

(b) Guess that, in this scenario, inflation and the output gap are linear in the cost push shock, $\hat{\pi}_t = \psi_{\pi u} u_t$ and $\hat{x}_t = \psi_{xu} u_t$ for two coefficients $\psi_{\pi u}$ and ψ_{xu}. Use the method of undetermined coefficients to solve for $\psi_{\pi u}$ and ψ_{xu}. Explain intuitively how inflation and the output gap respond to a cost push shock.

Now suppose the parameter values are $\kappa = 0.15$, $\beta = 0.95$, $\phi = 0.8$, $\sigma_u = 0.015$ and that the monetary authority's weight on inflation is $\lambda = 1$.

(c) Simulate a sequence of cost push shocks u_t of length $T = 250$ and use this to generate simulated sequences of inflation and the output gap. Does a scatterplot of inflation and the output gap reveal any relationship between inflation and the output gap? Are times of economic slack times of low inflation? Why or why not? Give intuition for your results

(d) Suppose you estimate a reduced form Phillip Curve relationship by OLS

$$\hat{\pi}_t = a + b\hat{x}_t + \varepsilon_t$$

Is the estimated relationship between inflation and the output gap upward or downward sloping? Why? How do the OLS coefficients a, b relate to the underlying parameters of the model? How do the OLS residuals ε_t relate to the underlying shocks? Explain.

SOLUTIONS:

(a) Let $\xi_t \equiv \beta \mathbb{E}_t \{\hat{\pi}_{t+1}\} + u_t$ denotes terms that the monetary authority takes as given. Substituting out $\hat{\pi}_t$ in the objective, the monetary authority chooses $\hat{x}_t$ to minimize

$$\frac{1}{2}\left(\hat{x}_t^2 + \lambda(\kappa \hat{x}_t + \xi_t)^2\right)$$

The first order condition for this problem can be written

$$\hat{x}_t = -\kappa\lambda\hat{\pi}_t$$

so that the monetary authority 'leans-against-the-wind' in the sense of reducing the output gap when inflation rises. Then plugging this into the new Keynesian Phillips curve gives

$$\hat{\pi}_t = \beta \mathbb{E}_t \{\hat{\pi}_{t+1}\} - \kappa^2\lambda\hat{\pi}_t + u_t$$

or

$$\hat{\pi}_t = \frac{\beta}{1+\kappa^2\lambda}\mathbb{E}_t \{\hat{\pi}_{t+1}\} + \frac{1}{1+\kappa^2\lambda}u_t$$

This is a single stochastic difference equation in $\hat{\pi}_t$ taking as given the process for the cost push shocks u_t.

(b) If $\hat{\pi}_t = \psi_{\pi u} u_t$ then $\mathbb{E}_t \{\hat{\pi}_{t+1}\} = \psi_{\pi u}\phi u_t$ so that on plugging these expressions into the stochastic difference equation for inflation we have

$$\psi_{\pi u} u_t = \frac{\beta}{1+\kappa^2\lambda}\psi_{\pi u}\phi u_t + \frac{1}{1+\kappa^2\lambda}u_t$$

Since this must hold for every u_t we have the restriction

$$\psi_{\pi u} = \frac{\beta}{1+\kappa^2\lambda}\psi_{\pi u}\phi + \frac{1}{1+\kappa^2\lambda}$$

and hence

$$\psi_{\pi u} = \frac{1}{(1-\phi\beta)+\kappa^2\lambda}$$

Since the discretionary policy is $\hat{x}_t = -\kappa\lambda\hat{\pi}_t$ we then have

$$\psi_{xu} = -\kappa\lambda\psi_{\pi u} = -\frac{\kappa\lambda}{(1-\phi\beta)+\kappa^2\lambda}$$

Hence a cost push shock $u_t > 0$ increases inflation $\hat{\pi}_t > 0$ and reduces the output gap $\hat{x}_t < 0$ (just like an adverse aggregate supply shock in a static AS-AD model).

(c)-(d) The results are shown in Figure 1 below. The scatterplot shows a pronounced negative correlation between economic activity as measured by the output gap $\hat{x}_t$ and inflation $\hat{\pi}_t$. Naively, we might take this to demonstrate the absence of a 'Phillips curve' or aggregate supply relationship like $\hat{\pi}_t = \kappa\hat{x}_t + \xi_t$ since that would be an upward sloping relationship between $\hat{x}_t$ and $\hat{\pi}_t$. The problem with this thinking is that both $\hat{x}_t$ and $\hat{\pi}_t$ are endogenous. In general, there is no particular reason to see either an 'AS curve' (or 'AD curve') in a reduced form scatterplot. What we see in the scatterplot depends on the relative importance of the underlying shocks. This model has *only* supply shocks and no demand shocks so what we see is a tracing out of the 'AD curve' implied by the optimal monetary policy.

More formally, since

$$\hat{\pi}_t = \frac{1}{(1-\phi\beta)+\kappa^2\lambda}u_t, \qquad \text{and} \qquad \hat{x}_t = -\frac{\kappa\lambda}{(1-\phi\beta)+\kappa^2\lambda}u_t$$

we have

$$\begin{aligned}
\text{Corr}[\hat{x}_t, \hat{\pi}_t] &= \frac{\text{Cov}[\hat{x}_t, \hat{\pi}_t]}{\sqrt{\text{Var}[\hat{x}_t]\text{Var}[\hat{\pi}_t]}} \\
&= \frac{\left(-\frac{\kappa\lambda}{(1-\phi\beta)+\kappa^2\lambda}\right)\left(\frac{1}{(1-\phi\beta)+\kappa^2\lambda}\right)\text{Var}[u_t]}{\left(\frac{\kappa\lambda}{(1-\phi\beta)+\kappa^2\lambda}\right)\left(\frac{1}{(1-\phi\beta)+\kappa^2\lambda}\right)\text{Var}[u_t]} \\
&= -1
\end{aligned}$$

Clearly the regression intercept $a = 0$ while the regression slope coefficient b is given by

$$b = \frac{\text{Cov}[\hat{x}_t, \hat{\pi}_t]}{\text{Var}[\hat{x}_t]} = \frac{\left(-\frac{\kappa\lambda}{(1-\phi\beta)+\kappa^2\lambda}\right)\left(\frac{1}{(1-\phi\beta)+\kappa^2\lambda}\right)\text{Var}[u_t]}{\left(\frac{\kappa\lambda}{(1-\phi\beta)+\kappa^2\lambda}\right)^2\text{Var}[u_t]} = -\frac{1}{\kappa\lambda}$$

That is, the regression identifies the monetary policy 'targeting rule' $\hat{x}_t = -\kappa\lambda\hat{\pi}_t$ not anything to do with the AS curve. Monetary policy is endogenously engineering a negative correlation between inflation and the output gap to offset the cost push shocks. The regression coefficient b captures the strength of that policy reaction. The regression residuals ε_t are identically zero independent of the actual structural shocks u_t.

So even though there is an 'AS curve' in the model, we don't see it in the reduced form relationship because there is no demand-side variation that could identify it. Please keep this example in mind when someone says that the lack of a reduced form relationship between inflation and unemployment proves there is no Phillip curve!

The same logic is pervasive in economics. If we do a scatterplot of prices and quantities, we don't generally expect to see either a supply curve or a demand curve but instead a cloud of points. The fact that we see a cloud of points does not invalidate the supply and demand curve analysis, but we we need disturbances that shift demand but not supply to identify the supply curve and disturbances that shift supply but not demand to identify the demand curve. No one would say that supply curves 'don't exist' simply because they cannot see a reduced form upward sloping relationship between price and quantity.

2. **Discretionary monetary policy in a liquidity trap.** Consider a continuous-time new Keynesian model with dynamic IS curve

$$\dot{x}(t) = \frac{1}{\sigma}(i(t) - \pi(t) - r^n(t))$$

and Phillips curve

$$\dot{\pi}(t) = \rho\pi(t) - \kappa x(t)$$

Here $x(t)$ denotes the log output gap, $\pi(t)$ the instantaneous inflation rate, $i(t)$ the nominal interest rate, and $r^n(t)$ the natural real rate. The parameters σ, ρ, κ have their usual meanings.

Suppose that the natural real rate follows

$$r^n(t) = \begin{cases} \underline{r} & t \in [0, T) \\ \\ \overline{r} & t \in [T, \infty) \end{cases} \qquad \text{with} \quad \underline{r} < 0 < \overline{r}$$

for some given horizon $T > 0$ with $\underline{r}$ sufficiently negative that the ZLB constraint on $i(t)$ is binding for $t \in [0, T)$. Suppose that for $r^n(t) = \overline{r} > 0$ it is possible to implement $i(t) = \overline{r}$ with a sufficiently reactive interest rate rule.

In this question you will study outcomes under monetary policy *discretion*.

(a) Explain, qualitatively, how inflation $\pi(t)$ and the output gap $x(t)$ are determined both during and after the liquidity trap under monetary policy discretion.

Now suppose the parameter values: $\sigma = 1$, $\kappa = 0.5$, and $\rho = 0.05$ per year with $\overline{r} = +\rho$, $\underline{r} = -\rho$ and $T = 2$ years.

(b) Solve for the time-path of $\pi(t)$ and $x(t)$ during and after the liquidity trap. Plot $\pi(t)$ and $x(t)$ against time and against each other. Explain your findings.

(c) How would your answers differ if instead $\kappa = 0.25$? or $\kappa = 0.75$? What if instead $T = 1$ years? or $T = 3$ years? And what about if $\sigma = 0.5$? or $\sigma = 2$? Give intuition for your answers.

SOLUTIONS:

(a) After the liquidity trap, i.e., for $t \in [T, \infty)$ monetary policy is able to implement $i(t) = \overline{r} > 0$ via a sufficiently reactive interest rate rule, for example

$$i(t) = \overline{r} + \phi_\pi \pi(t), \qquad \phi_\pi > 1$$

Under such a rule, we know that in the absence of shocks we simply have $\pi(t), x(t) = (0, 0)$ for all $t \in [T, \infty)$ and in particular have $\pi(T), x(T) = (0, 0)$. This then serves as a 'terminal condition' for the dynamics during the liquidity trap, i.e., for $t \in [0, T)$. During the liquidity trap we have $r^n(t) = \underline{r} < 0$ and $i(t) = 0$ so that the change in the output gap is

$$\dot{x}(t) = -\frac{1}{\sigma}(\pi(t) + \underline{r})$$

Thus $\dot{x}(t) > 0$ whenever $\pi(t) + \underline{r} < 0$ i.e., whenever $\pi(t) < -\underline{r}$. Figure 2 below shows the associated phase diagram in in (π, x) space. The $\dot{x}(t) = 0$ isocline is a vertical line at $\pi = -\underline{r}$ and divides the (π, x) plane into two halves, one to the left of $\pi = -\underline{r}$ with relatively low inflation where the output gap is increasing and another to the right of $\pi = -\underline{r}$ with relatively high inflation where the output gap is decreasing. Notice that on the left side where inflation is relatively low, the real interest rate $r(t) = i(t) - \pi(t) = -\pi(t)$ is relatively high — this is of course why the IS curve gives $\dot{x}(t) > 0$ in this region. Similarly we have from the new Keynesian Phillips curve that

$$\dot{\pi}(t) = \rho\pi(t) - \kappa x(t)$$

so that $\dot{\pi}(t) > 0$ whenever $\rho\pi(t) - \kappa x(t) > 0$ i.e., whenever $x(t) < (\rho/\kappa)\pi(t)$. In terms of the phase diagram, the $\dot{\pi}(t) = 0$ isocline is a straight line $x = (\rho/\kappa)\pi$ and again divides the (π, x) phase plane into two halves, one above the line $x = (\rho/\kappa)\pi$ where inflation is decreasing and the other below the line $x = (\rho/\kappa)\pi$ where inflation is increasing. These dynamics are illustrated in Figure 2 below. Notice that the intersection of the two isoclines picks out a pseudo 'steady state' $(\bar{\pi}, \bar{x}) = (-\underline{r}, -(\rho/\kappa)\underline{r})$ where neither inflation not the output gap are changing. This pseudo steady state vanishes when the liquidity trap is over (i.e., when $r^n(t)$ reverts to $\overline{r}$). Inspecting the phase diagram, we see that the only region where the dynamics can begin and converge to $x(T), \pi(T) = (0, 0)$ is in the southwest quadrant (to the right of the $\pi = -\underline{r}$ line, below the $x = (\rho/\kappa)\pi$ line) where both the output gap and inflation are increasing both from negative initial levels. In short, these dynamics imply that the economy begins with a recession $x(0) < 0$ and deflation $\pi(0) < 0$ that is then gradually alleviated as the economy approaches the end of the liquidity trap. In no other region of the phase plane do the dynamics approach $(0, 0)$ as $t \to T$.

(b) To solve for the time-path of inflation and the output gap, we first reduce the system to a second order differential equation in inflation and then solve that differential equation. To do this, begin by differentiating the new Keynesian Phillips curve to get

$$\ddot{\pi}(t) = \rho\dot{\pi}(t) - \kappa\dot{x}(t)$$

Then substitute in the IS curve to get

$$\ddot{\pi}(t) = \rho\dot{\pi}(t) + \frac{\kappa}{\sigma}(\pi(t) + \underline{r})$$

or

$$\ddot{\pi}(t) - \rho\dot{\pi}(t) - \frac{\kappa}{\sigma}\pi(t) = \frac{\kappa}{\sigma}\underline{r}$$

To solve this second order linear differential equation, we first solve the associated homogeneous equation

$$\ddot{\pi}(t) - \rho\dot{\pi}(t) - \frac{\kappa}{\sigma}\pi(t) = 0$$

The roots of this are given by the quadratic

$$r^2 - \rho r - \frac{\kappa}{\sigma} = 0$$

Since the trace is $r_1 + r_2 = \rho > 0$ and the determinant is $r_1 r_2 = -(\kappa/\sigma)$, we know that one root is negative and one root is positive. Since we have a given terminal condition $\pi(T), x(T) = (0, 0)$ we cannot disregard the dynamics associated with the unstable root (in other words, we don't need to worry about solutions blowing up as $t \to \infty$ since here we have finite $t \to T$). Hence solutions of the homogeneous equation have the form

$$\pi(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

for some constants to be determined (as part of the general solution, see below). We now need to find a particular solution. Since the exogenous term in the differential equation is a constant, it's natural to guess a constant solution say $\pi(t) = \bar{\pi}$ for which $\ddot{\pi}(t) = \dot{\pi}(t) = 0$. For this to be a solution we need

$$0 - \rho 0 - \frac{\kappa}{\sigma}\bar{\pi} = \frac{\kappa}{\sigma}\underline{r}$$

or

$$\bar{\pi} = -\underline{r} > 0$$

This is just the level of inflation associated with the pseudo steady state from part (a) above. The general solution therefore has the form

$$\pi(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \bar{\pi}$$

where r_1, r_2 are the roots of the quadratic, $\bar{\pi} = -\underline{r} > 0$ and c_1, c_2 are two constants to be determined. Notice that this general solution implies that initial inflation is simply

$$\pi(0) = c_1 + c_2 + \bar{\pi}$$

To determine the constants, we use the terminal conditions $\pi(T) = 0$ and $x(T) = 0$. The former implies

$$\pi(T) = c_1 e^{r_1 T} + c_2 e^{r_2 T} + \bar{\pi} = 0$$

This is one equation in the two unknowns c_1, c_2 given that we have determined $r_1, r_2, \bar{\pi}$ and that T is a parameter. Now if $\pi(T) = 0$ and $x(T) = 0$ we also know from the new Keynesian Phillips curve that $\dot{\pi}(T) = 0$ too. Differentiating the general solution with respect to t gives

$$\dot{\pi}(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$$

Hence for $t = T$ we have

$$\dot{\pi}(T) = c_1 r_1 e^{r_1 T} + c_2 r_2 e^{r_2 T} = 0$$

This constitutes a second equation in the two unknowns c_1, c_2. In short we have the linear system

$$\begin{pmatrix} e^{r_1 T} & e^{r_2 T} \\ r_1 e^{r_1 T} & r_2 e^{r_2 T} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \bar{\pi} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

to be solved for c_1, c_2. Doing the algebra gives

$$c_1 = +\frac{r_2}{r_1 - r_2}\, e^{-r_1 T}\, \bar{\pi}$$

and

$$c_2 = -\frac{r_1}{r_1 - r_2}\, e^{-r_2 T}\, \bar{\pi}$$

Plugging these back into the general solution and collecting terms

$$\pi(t) = \bar{\pi} - \frac{1}{r_1 - r_2} \left\{ r_1 e^{-r_2 (T-t)} - r_2 e^{-r_1 (T-t)} \right\} \bar{\pi}$$

so that indeed $\pi(t) \to 0$ as $t \to T$. The attached Matlab file `ps3_question2.m` implements this solution for the given parameter values. The roots work out to be $r_1 = 0.7325$ and $r_2 = -0.6825$ and $\bar{\pi} = \rho = 0.05$ so that for $T = 2$ the constants are $c_1 = -0.0056$ and $c_2 = -0.1014$. The initial level of deflation is then $\pi(0) = c_1 + c_2 + \bar{\pi} = -0.0569$ or about -5.69% on an annual basis. Finally we can recover the initial output gap from the new Keynesian Phillips curve and our solution for inflation

$$x(0) = \frac{\rho \pi(0) - \dot{\pi}(0)}{\kappa}$$

Since $\dot{\pi}(0) > 0$ in this part of the phase space, we know $x(0) < 0$ and indeed $x(0)$ lies below the line $x = (\rho/\kappa)\pi$. Evaluating the expression above gives $x(0) = -0.1359$, a severe recession with output about 13.59% below its natural level. The dynamics are shown in Figure 3 below. To construct this, I took a linearly spaced set of 200 discrete points to approximate the interval $[0, T]$ and evaluated the solution given above at those discrete points. The left hand panel shows $\pi(t), x(t)$ in the phase plane, the right hand panel shows $\pi(t)$ and $x(t)$ over time.

(c) If instead we have $\kappa = 0.25$ then prices are less flexible so that the initial output loss and deflation are smaller than in part (b) above. Specifically we now have $\pi(0) = -0.0263$ which is less deflation that the -0.0569 we had in (b) and similarly $x(0) = -0.1171$, a somewhat smaller output loss than the -0.1359 we had in (b). In the liquidity trap, more rigid prices reduce the output loss. This is because a low value of κ means that any given output loss $x(t) < 0$ creates less deflation $\pi(t) < 0$ via the Phillips curve so that real rates $r(t) = -\pi(t)$ are lower than they would otherwise be. Then from the IS curve we have that lower real rates implies lower output gap growth $\dot{x}(t)$ so that to reach $x(T) = 0$ in the same amount of time T we must start from a closer point away, i.e., from a $x(0)$ that is closer to 0. Similarly, if instead we have $\kappa = 0.75$ then prices are more flexible so that the initial output loss and deflation are greater than in part (b) above. Specifically we now have $\pi(0) = -0.0925$ and $x(0) = -0.1566$. In the liquidity trap, more flexible prices makes the output loss worse. This is because a high value of κ means that any given output loss $x(t) < 0$ creates more deflation $\pi(t) < 0$ via the Phillips curve so that real rates $r(t) = -\pi(t)$ are higher than they would otherwise be. Then from the IS curve we have that higher real rates implies higher output gap growth $\dot{x}(t)$ so that to reach $x(T) = 0$ in a fixed amount of time T we must start from further away, i.e., from lower $x(0)$.

If $T = 1$ then the liquidity trap is shorter and the initial output loss and deflation are again less than in part (b). Specifically we now have $\pi(0) = -0.0128$, less deflation than in (b), and similarly $x(0) = -0.0542$, a smaller output loss than in (b). This is because the eigenvalues of the system are still $r_1 = 0.7325$ and $r_2 = -0.6825$ (i.e., the speed of adjustment is independent of T) so that if we are to reach the same terminal point $(0, 0)$ travelling at the same speed but are to take a shorter interval of time $[0, T)$ to make that journey then we must be starting from a closer point (i.e., from initial conditions $\pi(0), x(0)$ that are closer to $0, 0$). Similarly, if instead we had $T = 3$ then the liquidity trap lasts longer and the initial output loss and deflation are again greater than in part (b). Specifically we now have $\pi(0) = -0.1533$ and $x(0) = -0.2852$. This is because the eigenvalues of the system are still $r_1 = 0.7325$ and $r_2 = -0.6825$ so that if we are to reach the same terminal point $(0, 0)$ travelling at the same speed but are to take a longer interval of time $[0, T)$ to make that journey then we must be starting from further away (i.e., from lower $\pi(0), x(0)$).

Finally, if $\sigma = 0.5$ then the intertemporal elasticity of substitution $1/\sigma$ is higher so that the output gap is more responsive to the real interest rate. Accordingly, any given deflation $\pi(t) < 0$ translates into a larger fall in output and hence more actual deflation. Specifically we now have $\pi(0) = -0.1334$, more deflation than in (b), and a whopping $x(0) = -0.3584$, a much larger fall in output than in (b). Similarly, if instead $\sigma = 2$ then the intertemporal elasticity of substitution $1/\sigma$ is lower so that the output gap is less responsive to the real interest rate and any given deflation $\pi(t) < 0$ translates into a smaller fall in output and hence less actual deflation. Specifically we now have $\pi(0) = -0.0263$ and $x(0) = -0.0585$.

3. **Unemployment fluctuations in a discrete time search model.** Let time be $t = 0, 1, 2, \ldots$. Risk neutral workers and firms have common time discount factor β and are matched according to a Cobb-Douglas matching function

$$F(u_t, v_t) = \bar{m}\, u_t^{1-\alpha} v_t^{\alpha}, \qquad \bar{m} > 0, \qquad 0 < \alpha < 1$$

where u_t denotes the unemployment rate and v_t the vacancy rate at time t. Let $\theta_t \equiv v_t/u_t$ denote labor market tightness and let $f(\theta_t)$ and $q(\theta_t)$ denote the job finding probability and vacancy filling probability respectively. If a match forms in period t then the worker and firm are able to start producing in period $t+1$. Within a match, the firm's productivity $z_t > 0$ follows an exogenous stochastic process with long-run mean $\bar{z} > 0$. The worker receives a wage w_t and the firm makes profits of $z_t - w_t$. Job matches are exogenously destroyed with probability $\delta \in (0, 1)$ per period. Firms can create jobs by posting vacancies with a per period cost $\kappa\bar{z} > 0$. When unemployed, workers receive constant flow utility $b\bar{z} \leq w_t$ from unemployment benefits.

The aggregate unemployment rate evolves according to

$$u_{t+1} - u_t = \delta(1 - u_t) - f(\theta_t)u_t$$

given some initial unemployment rate $u_0 > 0$.

(a) Let V_t and J_t denote the value to a firm of a vacancy and a filled job. These satisfy the discrete time Bellman equations

$$V_t = -\kappa\bar{z} + \beta\mathbb{E}_t\{\, q(\theta_t)J_{t+1} + (1 - q(\theta_t))V_{t+1} \,\}$$

$$J_t = z_t - w_t + \beta\mathbb{E}_t\{\, \delta V_{t+1} + (1 - \delta)J_{t+1} \,\}$$

Similarly let U_t and W_t denote the value to a worker of unemployment and employment. These satisfy the discrete time Bellman equations

$$U_t = b\bar{z} + \beta\mathbb{E}_t\{\, f(\theta_t)W_{t+1} + (1 - f(\theta_t))U_{t+1} \,\}$$

$$W_t = w_t + \beta\mathbb{E}_t\{\, \delta U_{t+1} + (1 - \delta)W_{t+1} \,\}$$

Provide an intuitive interpretation of these four Bellman equation.

Now suppose the wage is determined by Nash-Bargaining so that in equilibrium the worker's surplus is a constant fraction $\lambda \in (0, 1)$ of the total match surplus

$$W_t - U_t = \lambda S_t, \qquad S_t \equiv W_t - U_t + J_t - V_t$$

Suppose also that free-entry drives the value of a vacancy to $V_t = 0$.

(b) Explain how to solve for the non-stochastic steady state values of labor market tightness, wages, the unemployment rate and the vacancy rate in this setting. Explain intuitively the effects of a change in average productivity $\bar{z}$.

Now suppose that productivity follows a stationary AR(1) process in logs

$$\log z_{t+1} = (1 - \phi)\log\bar{z} + \phi\log z_t + \varepsilon_{t+1}, \qquad 0 < \phi < 1$$

where the innovations ε_t are IID $N(0, \sigma_\varepsilon^2)$.

Let the parameter values be $\alpha = 0.5$, $\beta = 1/1.02$, $\delta = 0.04$, $\phi = 0.95$, $\sigma_\varepsilon = 0.01$, $\bar{z} = 1$, $\bar{m} = 0.5$, $b = 0.4$, $\lambda = 0.5$, and $\kappa = 0.28$.

(c) Solve for the non-stochastic steady state values of labor market tightness, wages, the unemployment rate and the vacancy rate.

(d) Use DYNARE to solve the model. Suppose the economy is at steady state and that at $t = 0$ there is a 1% innovation to productivity, i.e., $\varepsilon_0 = 0.01$. Use DYNARE to calculate and plot the impulse response functions for the log-deviations of labor market tightness, wages, the unemployment rate and the vacancy rate for $T = 100$ periods after the shock.

(e) Simulate a sequence of productivity $\hat{z}_t$ of length $T = 500$ and use this to generate simulated sequences of labor market tightness, wages, the unemployment rate and the vacancy rate. Which of these variables move most closely together? Which is most volatile? Explain.

SOLUTIONS:

(a) For a firm, the value of having a filled job J_t is given by the flow profit $z_t - w_t$ plus the expected discounted value of its situation next period: with exogenous probability δ the job is destroyed and the firm switches to having an open vacancy with value V_{t+1} and with probability $1-\delta$ the job is not destroyed and the firm obtains value J_{t+1}. Likewise the firm's value of having a vacancy V_t is given by the flow cost of keeping a vacancy open $-\kappa\bar{z}$ plus the expected discounted value of its situation next period: with endogenous probability $q(\theta_t)$ there is a match and the vacancy is filled so that the firm switches to having a filled job with value J_{t+1} and with probability $1 - q(\theta_t)$ the vacancy is not filled and the firm obtains value V_{t+1}. For a worker, the value of having a job W_t is given by the flow wage w_t plus the expected discounted value of its situation next period: with probability δ the job is destroyed and the workers switches to being unemployed with value U_{t+1} and with probability $1-\delta$ the job is not destroyed and the worker obtains value W_{t+1}. Likewise the worker's value of being unemployed U_t is given by the flow unemployment benefits $b\bar{z}$ plus the expected discounted value of its situation next period: with probability $f(\theta_t)$ there is a match so that the worker switches to having a job with value W_{t+1} and with probability $1 - f(\theta_t)$ the worker remains unemployed and obtains value U_{t+1}.

(b) In non-stochastic steady state we can write the Bellman equations for the firm

$$V = -\kappa\bar{z} + \beta\{q(\theta)J + (1 - q(\theta))V\}$$

$$J = \bar{z} - w + \beta\{\delta V + (1 - \delta)J\}$$

and for the worker

$$U = b\bar{z} + \beta\{f(\theta)W + (1 - f(\theta))U\}$$

$$W = w + \beta\{\delta U + (1 - \delta)W\}$$

And from Nash-Bargaining

$$W - U = \frac{\lambda}{1-\lambda}(J - V)$$

The firm's Bellman equation for a filled job gives

$$J = \frac{\bar{z} - w + \beta\delta V}{1 - \beta(1-\delta)}$$

and since $V = 0$ from free-entry, we also have

$$J = \frac{\bar{z} - w}{1 - \beta(1-\delta)} = \frac{\kappa\bar{z}}{\beta q(\theta)}$$

Rearranging this gives

$$w = \bar{z} - \frac{1-\beta(1-\delta)}{\beta}\frac{\kappa\bar{z}}{q(\theta)}$$

Now observe that if we define the discount rate $r \equiv \frac{1}{\beta} - 1$ we get an expression familiar from the continuous time setup

$$\boxed{w = \bar{z} - (r+\delta)\frac{\kappa\bar{z}}{q(\theta)}}$$

This is the 'marginal productivity condition,' a downward sloping relationship between θ and w. Turning now to the worker side of things, from Nash-Bargaining

$$W - U = \frac{\lambda}{1-\lambda}(J - V) = \frac{\lambda}{1-\lambda}\frac{\kappa\bar{z}}{\beta q(\theta)}$$

And the worker's Bellman equation for unemployment gives

$$(1-\beta)U = b\bar{z} + \beta f(\theta)\frac{\lambda}{1-\lambda}J = b\bar{z} + \frac{\lambda}{1-\lambda}\kappa\bar{z}\theta$$

since $f(\theta) = \theta q(\theta)$. Then from the worker's Bellman equation for employment

$$W = \frac{w + \beta\delta U}{1-\beta(1-\delta)}$$

so that

$$W - U = \frac{w - (1-\beta)U}{1-\beta(1-\delta)}$$

Using Nash-Bargaining again

$$\frac{w-(1-\beta)U}{1-\beta(1-\delta)} = \frac{\lambda}{1-\lambda}J = \frac{\lambda}{1-\lambda}\frac{\bar{z}-w}{1-\beta(1-\delta)}$$

Hence

$$w - (1-\beta)U = \frac{\lambda}{1-\lambda}(\bar{z} - w)$$

So that on using the expression for $(1-\beta)U$ above

$$\boxed{w = (1-\lambda)b\bar{z} + \lambda(1+\kappa\theta)\bar{z}}$$

This is the 'wage curve,' an upward sloping relationship between θ and w. Together, the wage curve and the marginal productivity condition are two equations that we can solve for the steady state values $\bar{w}, \bar{\theta}$. Given labor market tightness determined in this way, we can then back out the unemployment rate $\bar{u}$ from the Beveridge curve

$$\bar{u} = \frac{\delta}{\delta + f(\bar{\theta})}$$

and back out vacancies $\bar{v}$ from $\bar{v} = \bar{\theta}\bar{u}$. Notice that in this setting benefits $b\bar{z}$ and vacancy costs $\kappa\bar{z}$ are both proportional to $\bar{z}$. This implies that an increase in $\bar{z}$ shifts up both the marginal productivity condition and the wage curve proportionately so that labor market tightness is unchanged while the wage increases one-for-one with $\bar{z}$. Since labor market tightness is unchanged, from the Beveridge curve unemployment is unchanged too. We can see this more formally by writing the intersection of the marginal productivity condition and wage curve as

$$\frac{\bar{w}}{\bar{z}} = 1 - (r+\delta)\frac{\kappa}{q(\bar{\theta})} = (1-\lambda)b + \lambda(1+\kappa\bar{\theta}) \tag{$*$}$$

which determines steady state $\bar{\theta}$ independent of $\bar{z}$ and implies $\bar{w}$ is proportional to $\bar{z}$. In short, in this setting an increase in average labor productivity has no long-run effect on labor market tightness and hence no long-run effect on unemployment or vacancies and simply leads to a proportionate increase in the long-run wage.

(c) With the given parameter values, solving ($*$) for steady state labor market tightness gives $\bar{\theta} = 1.8192$ which implies the steady state wage $\bar{w} = 0.9547$. This means that a bit over 95% of the average product of labor is paid out to workers, leaving flow profits of $\bar{z} - \bar{w} = 0.0453$ for the firm, a profit rate a bit under 5% per period. Given this value of labor market tightness we get a job finding probability of $f(\bar{\theta}) = 0.6744$ so that, per period, about 67% of unemployed workers find a job. Then from the Beveridge curve we get $\bar{u} = \delta/(\delta + f(\bar{\theta})) = 0.04/0.7144 = 0.0560$, a steady state unemployment rate of 5.6%. This also implies that steady state vacancies are $\bar{v} = \bar{\theta}\bar{u} = 0.1019$.

(d) We solve for the following log-linear approximation

$$\begin{aligned}
\hat{u}_{t+1} &= \psi_{uu}\hat{u}_t + \psi_{uz}\hat{z}_t \\
\hat{w}_t &= \psi_{wu}\hat{u}_t + \psi_{wz}\hat{z}_t \\
\hat{\theta}_t &= \psi_{\theta u}\hat{u}_t + \psi_{\theta z}\hat{z}_t
\end{aligned}$$

The attached Dynare file `ps3_question3.mod` solves the model with the given parameters. From Dynare we get

```
POLICY AND TRANSITION FUNCTIONS
                    w           theta              u               z
u(-1)       -0.129006       -0.842853      -0.481588        0
z(-1)        0.809194        2.131194      -1.549122        0.950000
e            0.851783        2.243362      -1.630655        1.000000
```

so that, for example, $\psi_{uu} = -0.481588$ and $\psi_{uz} = -1.630655$. Figure 4 below shows the impulse response functions for the key endogenous variables to a one standard deviation shock to productivity. On impact, labor market tightness and the wage rise (indeed the response of the wage is almost identical to the response of productivity). With rising labor market tightness, the job finding rate (not shown) rises and unemployment falls. Notice that the negative serial correlation coefficient for unemployment $\psi_{uu} < 0$ imparts some mild short-run negative serial correlation to the dynamics. In the longer run, these wiggles are sufficiently dampened that the positive serial correlation from the exogenous productivity shock dominates and we get more familiar looking responses.

(e) From Dynare we get

```
THEORETICAL MOMENTS
VARIABLE        MEAN    STD. DEV.   VARIANCE
w            -0.0167     0.0317     0.0010
theta         1.9967     0.1011     0.0102
u            -3.9581     0.0364     0.0013
z             0.0000     0.0320     0.0010
```

Figure 5 below shows the simulated time series. Clearly labor market tightness is the most volatile while wages, unemployment, and productivity are roughly equally volatile. Labor market tightness, wages, and productivity are nearly perfectly positively correlated with each other. Unemployment is nearly perfectly negatively correlated with the others.

Figure 1: Scatter of inflation and output gap under optimal monetary policy

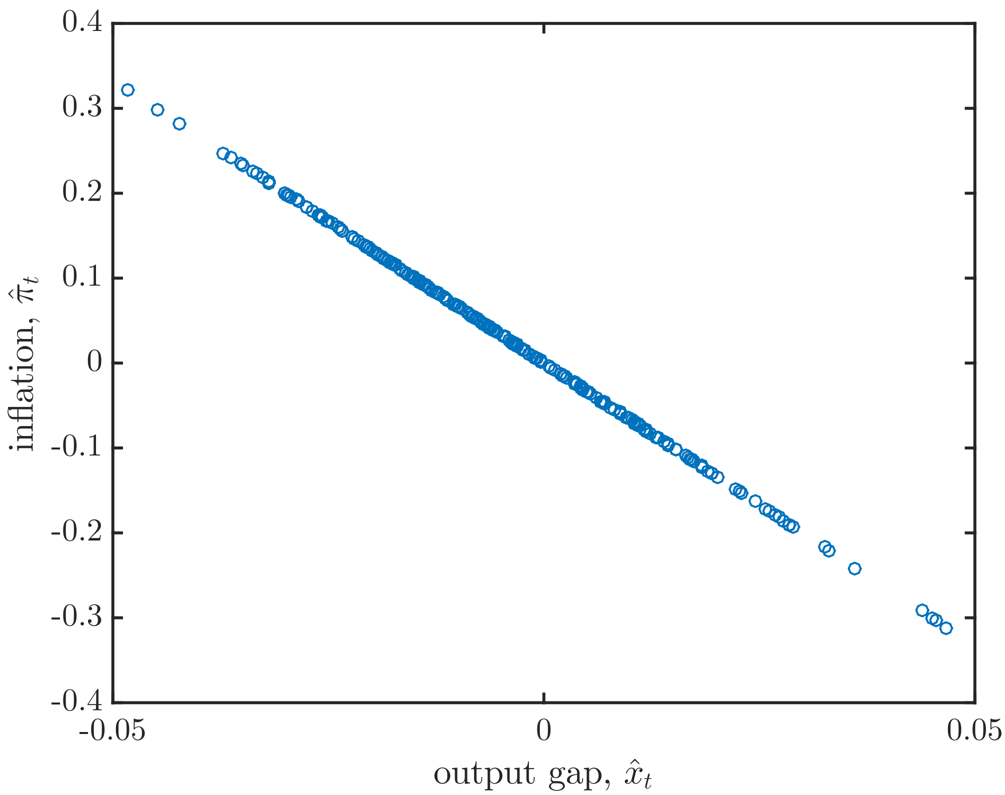

Figure 2: Phase diagram for discretionary monetary policy in a liquidity trap

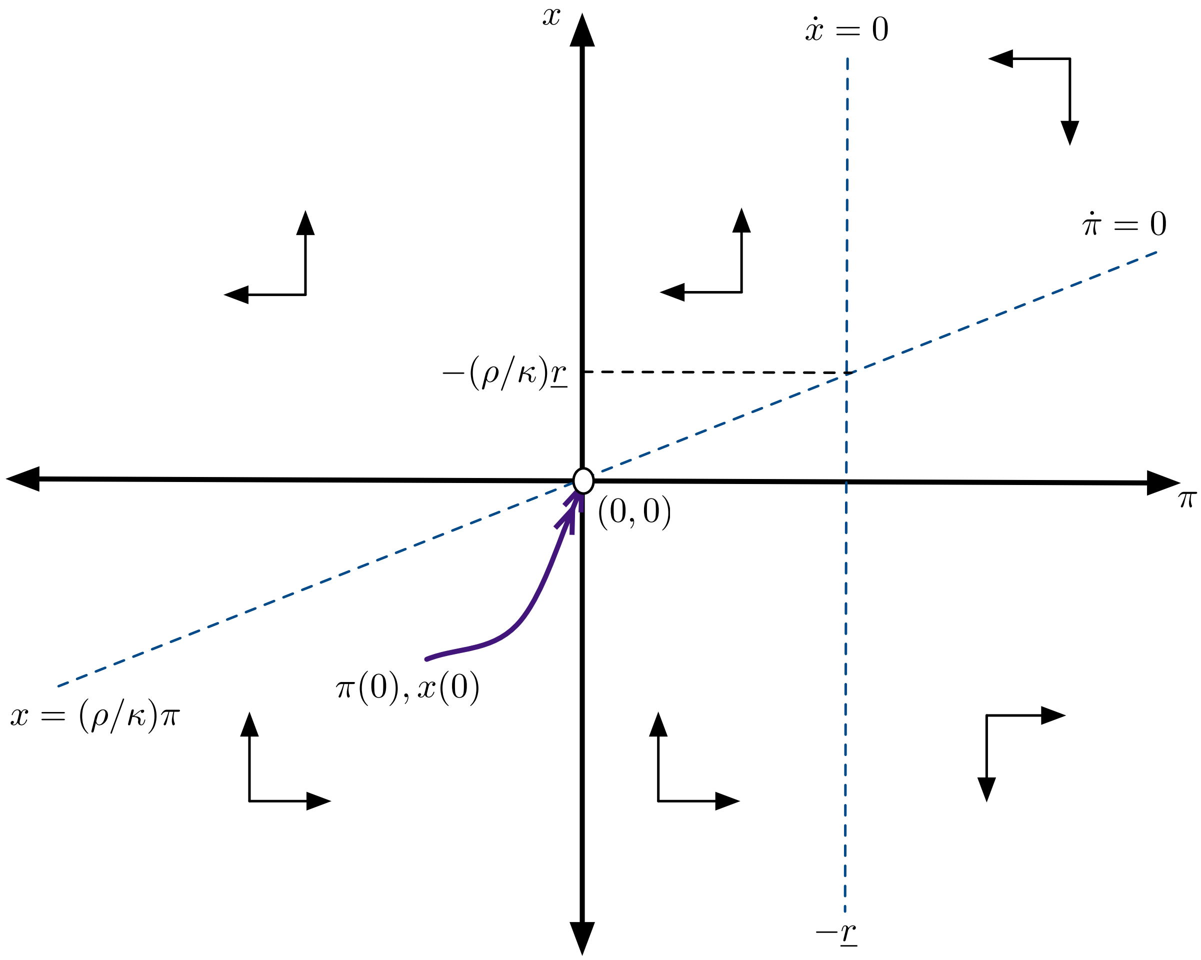

Figure 3: Impulse responses for discretionary monetary policy in a liquidity trap

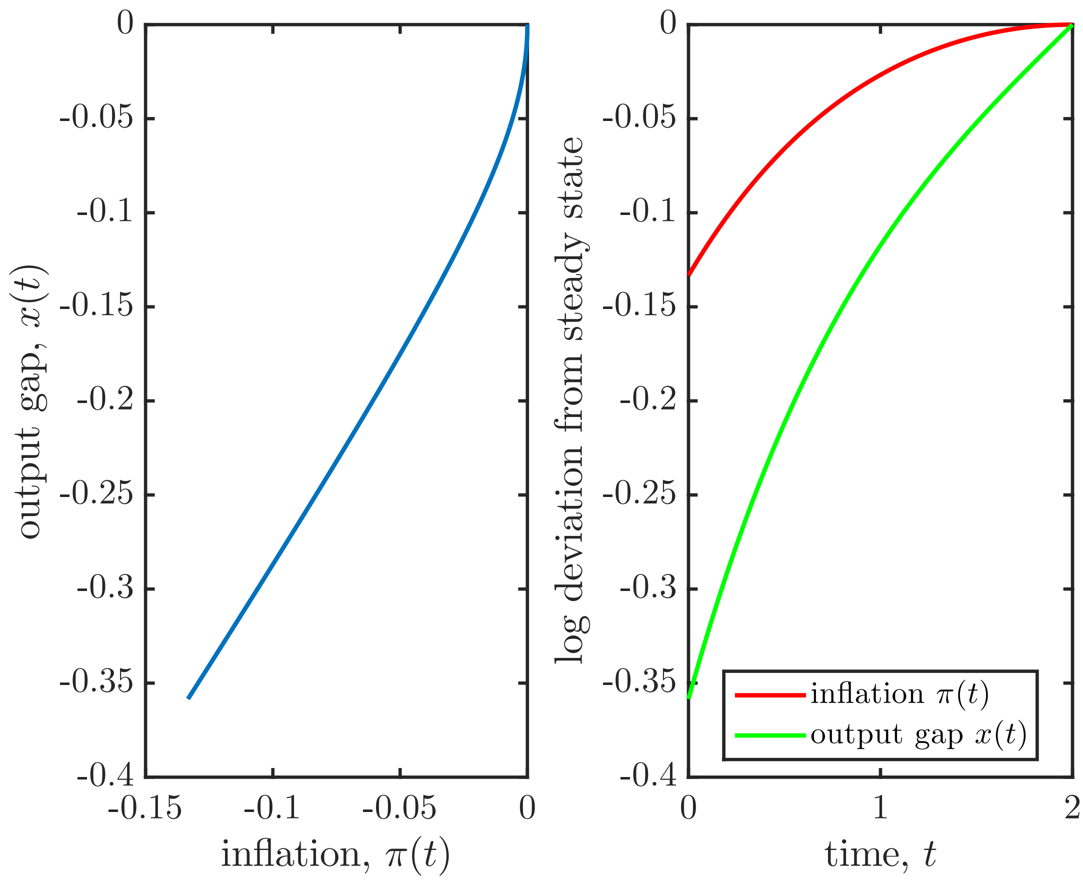

Figure 4: Response to productivity shock in search model

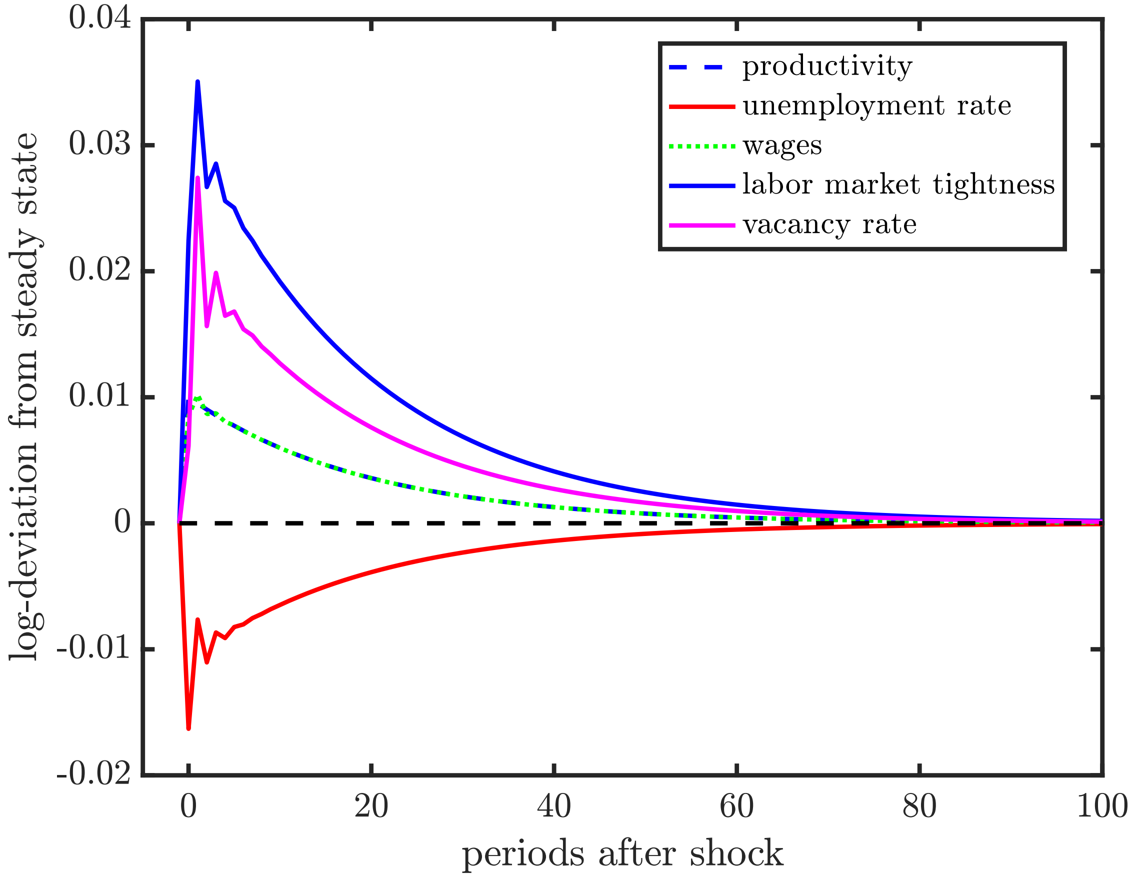

Figure 5: Labor market fluctuations in search model