Advanced Macroeconomics
Problem Set #2: Solutions

1. **Automation in a growth model.** Suppose a final good Y is produced by *perfectly competitive* firms using a Cobb-Douglas bundle of tasks

$$Y_t = \exp\left(\int_{N-1}^{N} \log y_t(i)\, di\right)$$

for some given interval $[N-1, N]$. All tasks can be done by labor, but some tasks can be done by labor or capital. In particular, there is a threshold task I such that the production technology for tasks $i > I$ is

$$y_t(i) = a_l l_t(i), \qquad i > I$$

while the production technology for tasks $i \leq I$ is

$$y_t(i) = a_k k_t(i) + a_l l_t(i), \qquad i \leq I$$

Each task is produced under perfectly competitive conditions taking as given the wage rate W_t and the rental rate R_t. To simplify the analysis, we tentatively suppose that W_t, R_t are such that

$$\frac{R_t}{a_k} < \frac{W_t}{a_l} \tag{$*$}$$

There are L identical households each of which supplies one unit of labor and seeks to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \qquad 0 < \beta < 1$$

subject to

$$c_t + k_{t+1} = W_t + (R_t + 1 - \delta)k_t \qquad 0 < \delta < 1$$

Let $C_t = c_t L$, $K_t = k_t L$ and L denote aggregate consumption, capital, and labor. In equilibrium the factor markets clear with $K_t = \int k_t(i)\, di$ and $L = \int l_t(i)\, di$.

(a) Let $Y = F(K, L)$ denote the aggregate production function, i.e., the amount of final output that the economy produces with aggregate capital K and labor L. Derive the aggregate production function for this economy.

(b) Show that in order for condition $(*)$ to hold the aggregate capital stock K_t must exceed a certain threshold

$$K_t > \hat{K}$$

Provide a formula for this threshold $\hat{K}$ in terms of the underlying parameters of the model.

(c) Solve for the steady state values of aggregate consumption, capital, output and the wage and rental rate in terms of model parameters. Is condition $(*)$ always satisfied in steady state? Explain.

(d) Let the parameter values be $N = 1$, $L = 1$, $a_l = 0.1$, $a_k = 0.2$, $\beta = 1/1.05$, $\delta = 0.05$. For each of the following grid of values

$$I \in \{0.25, 0.26, 0.27, ..., 0.49, 0.50\}$$

calculate and plot the steady state values of aggregate consumption, capital, output and wages. Does more automation increase output? Does more automation decrease wages? What is the role of capital accumulation? Explain your findings.

(e) How if at all do your answers to part (d) change if $\beta = 1/1.03$? Or if $\beta = 1/1.01$? Explain.

Solutions:

(a) From the Cobb-Douglas demand system the quantity demanded of each task is

$$y_t(i) = \frac{Y_t}{p_t(i)}$$

And since each task is produced under perfectly competitive conditions the price $p_t(i)$ is equal to marginal cost $\mathsf{mc}_t(i)$. Tasks $i \leq I$ are produced with capital and have marginal cost $\mathsf{mc}_t(i) = R_t/a_k$. Tasks $i > I$ are produced with labor and have marginal cost $\mathsf{mc}_t(i) = W_t/a_l$. This implies the demand for capital is

$$k_t(i) = \frac{y_t(i)}{a_k} = \frac{\frac{Y_t}{p_t(i)}}{a_k} = \frac{\frac{Y_t}{\mathsf{mc}_t(i)}}{a_k} = \frac{\frac{Y_t}{R_t/a_k}}{a_k} = \frac{Y_t}{R_t}, \qquad i \leq I$$

(with $k_t(i) = 0$ for $i > I$) and the demand for labor is

$$l_t(i) = \frac{y_t(i)}{a_l} = \frac{\frac{Y_t}{p_t(i)}}{a_l} = \frac{\frac{Y_t}{\mathsf{mc}_t(i)}}{a_l} = \frac{\frac{Y_t}{W_t/a_l}}{a_l} = \frac{Y_t}{W_t}, \qquad i > I$$

(with $l_t(i) = 0$ for $i \leq I$). The market for capital clears when

$$K_t = \int_{N-1}^{N} k_t(i)\, di = \int_{N-1}^{I} \frac{Y_t}{R_t}\, di = (I - (N-1))\frac{Y_t}{R_t}$$

The market for labor clears when

$$L = \int_{N-1}^{N} l_t(i)\, di = \int_{I}^{N} \frac{Y_t}{W_t}\, di = (N - I)\frac{Y_t}{W_t}$$

The factor income shares are therefore

$$s_K \equiv \frac{R_t K_t}{Y_t} = (I - (N-1))$$

and

$$s_L \equiv \frac{W_t L}{Y_t} = (N - I)$$

We can then express output for each task as

$$y_t(i) = \begin{cases} a_k \dfrac{K_t}{s_K} & i \leq I \\ \\ a_l \dfrac{L}{s_L} & i > I \end{cases}$$

Aggregate output is then given by

$$\begin{aligned} \log Y_t = \int_{N-1}^{N} \log y_t(i)\, di &= \int_{N-1}^{I} \log\left(a_k \frac{K_t}{s_K}\right) di + \int_{I}^{N} \log\left(a_l \frac{L}{s_L}\right) di \\ &= (I-(N-1)) \log\left(a_k \frac{K_t}{s_K}\right) + (N-I) \log\left(a_l \frac{L}{s_L}\right) \\ &= s_K \log\left(a_k \frac{K_t}{s_K}\right) + s_L \log\left(a_l \frac{L}{s_L}\right) \\ &= \log\left(\left(\frac{a_k}{s_K}\right)^{s_K} \left(\frac{a_l}{s_L}\right)^{s_L}\right) + s_K \log K_t + s_L \log L \end{aligned}$$

so that we can write the aggregate production function as

$$Y_t = F(K_t, L)$$

where

$$F(K, L) \equiv \left(\frac{a_k}{s_K}\right)^{s_K} \left(\frac{a_l}{s_L}\right)^{s_L} K^{s_K} L^{s_L}$$

where $s_K = (I - (N-1))$ and $s_L = (N - I)$ and $s_K + s_L = 1$.

(b) Condition $(*)$ is satisfied when

$$\frac{R_t}{a_k} < \frac{W_t}{a_l}$$

But we have just seen that $s_K = R_t K_t / Y_t$ and $s_L = W_t L / Y_t$ so $R_t = s_K Y_t / K_t$ and $W_t = s_L Y_t / L$ so we need

$$\frac{s_K Y_t / K_t}{a_k} < \frac{s_L Y_t / L}{a_l}$$

or equivalently

$$K_t > \frac{s_K}{s_L} \frac{a_l}{a_k} L \equiv \hat{K}$$

where again $s_K = (I - (N-1))$ and $s_L = (N - I)$. The point being that when the economy has accumulated 'enough' capital, $K_t > \hat{K}$, capital will be sufficiently abundant and the factor price of capital sufficiently low that it will be optimal to use capital to produce any task that can be produced with capital — i.e., that automation will lead to labor displacement.

(c) The key first order conditions for each household include their consumption Euler equation

$$u'(c_t) = \beta u'(c_{t+1})(R_{t+1} + 1 - \delta)$$

and budget constraint

$$c_t + k_{t+1} = W_t + (R_t + 1 - \delta) k_t$$

Aggregate consumption is $C_t = c_t L$, aggregate capital is $K_t = k_t L$ etc. Aggregating the household budget constraints gives

$$C_t + K_{t+1} = W_t L + (R_t + 1 - \delta) K_t$$

From the consumption Euler equation in steady state we have

$$1 = \beta(\bar{R} + 1 - \delta)$$

or

$$\bar{R} = \rho + \delta, \qquad \rho \equiv \frac{1}{\beta} - 1$$

Hence the steady-state capital/output ratio is

$$\frac{\bar{K}}{\bar{Y}} = \frac{s_K}{\bar{R}} = \frac{s_K}{\rho + \delta}$$

From the household budget constraints in steady state

$$\bar{C} + \bar{K} = \bar{W} L + (\bar{R} + 1 - \delta)\bar{K}$$

or

$$\bar{C} + \delta \bar{K} = \bar{W} L + \bar{R}\bar{K} = \bar{Y}$$

Hence the steady-state consumption/output ratio is

$$\frac{\bar{C}}{\bar{Y}} = 1 - \delta \frac{\bar{K}}{\bar{Y}} = 1 - \delta \frac{s_K}{\rho + \delta}$$

We now need to determine the actual level of output. To do this, write the aggregate production function

$$Y = Z\, K^{s_K}\, L^{1-s_K}, \qquad Z \equiv \left(\frac{a_k}{s_K}\right)^{s_K} \left(\frac{a_l}{s_L}\right)^{s_L}$$

But this means

$$1 = Z \left(\frac{K}{Y}\right)^{s_K} \left(\frac{L}{Y}\right)^{1-s_K}$$

hence steady-state output per worker is

$$\frac{\bar{Y}}{L} = Z^{\frac{1}{1-s_K}} \left(\frac{\bar{K}}{\bar{Y}}\right)^{\frac{s_K}{1-s_K}} = Z^{\frac{1}{1-s_K}} \left(\frac{s_K}{\rho + \delta}\right)^{\frac{s_K}{1-s_K}}$$

So we have

$$\bar{Y} = Z^{\frac{1}{1-s_K}} \left(\frac{s_K}{\rho + \delta}\right)^{\frac{s_K}{1-s_K}} L$$

and hence we now have

$$\bar{K} = \frac{s_K}{\rho + \delta}\, \bar{Y}$$

$$\bar{C} = \left(1 - \delta \frac{s_K}{\rho + \delta}\right) \bar{Y}$$

Finally, the wage is given by

$$\bar{W} = s_L \frac{\bar{Y}}{L}$$

where again $s_K = I - (N-1)$ and $s_L = (N-I)$ with $s_K + s_L = 1$.
From part (b) that condition $(*)$ is satisfied in steady state if and only if

$$\bar{K} > \hat{K} \equiv \frac{s_K}{s_L}\frac{a_l}{a_k} L$$

The steady-state capital/labor ratio is

$$\frac{\bar{K}}{L} = \left(\frac{s_K Z}{\rho+\delta}\right)^{\frac{1}{1-s_K}}$$

So condition $(*)$ is satisfied if and only if

$$\left(\frac{s_K Z}{\rho+\delta}\right)^{\frac{1}{1-s_K}} > \frac{s_K}{s_L}\frac{a_l}{a_k}$$

But now recall that Z is shorthand for

$$Z \equiv \left(\frac{a_k}{s_K}\right)^{s_K}\left(\frac{a_l}{s_L}\right)^{s_L}$$

So our condition is

$$\left(\frac{s_K}{\rho+\delta}\left(\frac{a_k}{s_K}\right)^{s_K}\left(\frac{a_l}{s_L}\right)^{s_L}\right)^{\frac{1}{1-s_K}} > \frac{s_K}{s_L}\frac{a_l}{a_k}$$

Since $s_L = 1 - s_K$ this is equivalent to

$$\left(\frac{s_K}{\rho+\delta}\left(\frac{a_k}{s_K}\right)^{s_K}\right)^{\frac{1}{1-s_K}} > \frac{s_K}{a_k}$$

or

$$\frac{s_K}{\rho+\delta}\left(\frac{a_k}{s_K}\right)^{s_K} > \left(\frac{s_K}{a_k}\right)^{1-s_K}$$

which simplifies to

$$\frac{1}{\rho+\delta} > \frac{1}{a_k}$$

or

$$\boxed{\rho + \delta < a_k}$$

In short, condition $(*)$ is *not* always satisfied in steady state. Condition $(*)$ is satisfied when the underlying fundamentals of the economy are conducive to capital accumulation, namely households are sufficiently patient (low discount rate ρ) and capital is sufficiently productive (high capital productivity a_k, low depreciation rate δ) so that $\bar{R} = \rho + \delta < a_k$.

(d)-(e) First notice that with these parameter values $\bar{R} = \rho + \delta = 0.05 + 0.05 = 0.1$ (10% per year, say) which is less than $a_k = 0.2$ so condition $(*)$ is satisfied. The comparison across different steady states for different levels of the automation threshold I are shown in Figure 1 below. The steady-state (i.e., 'long-run') values of capital, consumption, output, and the real wage are all increasing in I. In this sense, at least in the long run, automation increases output and wages (though labor's *share* of income $s_L = N - I$ mechanically decreases). In other words, wages do increase but not as much as labor productivity (output per worker).

Figure 2 below repeats these calculations for different time discount factors. For each I, the more patient the economy (higher is β) the higher is the level of steady-state capital. Graphically, each curve shifts up as we consider higher values of β.

This demonstrates a key difference between this model and the version we covered in class. Here there is endogenous capital accumulation so as the economy becomes more productive it also accumulates more capital (and this effect is larger the more patient people are) and so is able to produce more which acts as an additional source of labor demand and hence an additional force that tends to drive up long-run real wages. By contrast, in the version discussed in class the calculation is entirely static and there is simply a given amount of capital and labor to be deployed. While the effect of automation on wages can still be decomposed into a 'displacement effect' and a 'labor productivity effect', the (long-run) labor-productivity effect is here stronger because it is amplified by the effects of capital accumulation.

2. **Markups in a business cycle model.** Consider a real business cycle model where L identical households seek to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big(\log c_t - \frac{l_t^{1+\varphi}}{1+\varphi} \Big), \qquad 0 < \beta < 1, \quad \varphi > 0$$

subject to the budget constraints

$$c_t + k_{t+1} = W_t l_t + (R_t + 1 - \delta) k_t + \pi_t, \qquad 0 < \delta < 1$$

where π_t denotes lump-sum profits paid out by firms.

Final output Y_t is produced by *perfectly competitive* firms using a CES bundle of intermediates

$$Y_t = \left(\int_0^1 y_t(i)^{1/\mu} \, di \right)^{\mu}, \qquad \mu > 1$$

The final good firms buy intermediate goods at prices $p_t(i)$ from intermediate producers $i \in [0, 1]$. The intermediate producers are *monopolistically competitive* and choose prices $p_t(i)$ and output $y_t(i)$ to maximize profits understanding their market power.

Intermediate producers have the Cobb-Douglas production function

$$y_t(i) = z_t k_t(i)^{\alpha} l_t(i)^{1-\alpha}, \qquad 0 < \alpha < 1$$

and take the economy-wide rental rate R_t and wage rate W_t as given. The exogenous stochastic process for productivity z_t is common to all firms.

Let $C_t = c_t L$, $K_t = k_t L$ and $L_t = l_t L$ denote aggregate consumption, capital, and employment. In equilibrium the factor markets clear with $K_t = \int k_t(i) \, di$ and $L_t = \int l_t(i) \, di$.

(a) Let $\mathsf{TC}_t(y)$ denote the total cost function of each intermediate producer. Show that the total cost function is linear in output

$$\mathsf{TC}_t(y) = \mathsf{mc}_t \, y$$

for some marginal cost mc_t. Derive an expression for marginal cost mc_t in terms of the factor prices W_t, R_t and productivity z_t. Show that intermediate producers set prices that are a *markup* over marginal cost and that this markup is equal to the parameter μ.

(b) Now consider a *symmetric equilibrium* where all intermediate producers set the same price $p_t(i) = p_t$. Derive the key conditions that allow you to show how consumption, capital and employment are determined in this equilibrium. Also explain how prices, the wage rate, rental rate of capital, and profits are determined.

(c) Solve for the non-stochastic steady-state values of consumption, capital and employment in terms of model parameters. Solve also for the steady-state values of producer prices, the wage rate, rental rate of capital, and profits.

(d) Suppose the economy is in the steady state you found in (c). Then suddenly there is a permanent increase in producer market power such that the markup increases permanently from μ to $\mu' > \mu$. Explain the long run responses of consumption, capital, employment, the wage rate, rental rate, and profits in response to this permanent rise in markups.

Now suppose productivity and markups follow independent stationary AR(1) processes in logs

$$\log z_{t+1} = (1-\phi_z)\log \bar{z} + \phi_z \log z_t + \varepsilon_{z,t+1}, \qquad 0 < \phi_z < 1$$

where the innovations $\varepsilon_{z,t}$ are IID $N(0, \sigma^2_{\varepsilon,z})$, and

$$\log \mu_{t+1} = (1-\phi_\mu)\log \bar{\mu} + \phi_\mu \log \mu_t + \varepsilon_{\mu,t+1}, \qquad 0 < \phi_\mu < 1$$

where the innovations $\varepsilon_{\mu,t}$ are IID $N(0, \sigma^2_{\varepsilon,\mu})$.

Let the parameter values be $\alpha = 0.3$, $\beta = 1/1.01$, $\delta = 0.02$, $\varphi = 1$, $\bar{z} = 1$, $\bar{\mu} = 1.15$, with common persistence $\phi_z = \phi_\mu = 0.95$ and innovation standard deviations $\sigma_{\varepsilon,z} = \sigma_{\varepsilon,\mu} = 0.01$.

(e) Use Dynare to solve the model. Use Dynare to calculate and plot the impulse response functions for the log-deviations of consumption, investment, output, employment, the wage rate, rental rate, and profits in response to both (i) a one standard deviation productivity shock, and (ii) a one standard deviation markup shock. How do the dynamic responses of the economy to these shocks compare? Give as much intuition as you can for your findings. Compare the dynamics of the economy in response to this transitory markup shock to the long-run effect of a permanent change in markups as in part (d) above.

(f) Use Dynare to calculate the standard deviations and cross-correlations of the log-deviations of consumption, investment, output, employment, the wage rate, rental rate, and profits conditional on (i) only productivity shocks, (ii) only markup shocks, and (iii) both shocks together. Explain your findings.

Solutions:

(a) **Cost function.** The cost function for each producer is defined by

$$\mathsf{TC}(y) \equiv \min_{k,l} \left[Rk + Wl \;\middle|\; zk^\alpha l^{1-\alpha} = y \right]$$

The Lagrangian for this minimization problem is

$$L = Rk + Wl + \lambda(y - zk^\alpha l^{1-\alpha})$$

which has the first order conditions

$$R = \lambda \alpha z k^{\alpha-1} l^{1-\alpha}$$

and

$$W = \lambda(1-\alpha) z k^{\alpha} l^{-\alpha}$$

Hence

$$Rk = \lambda \alpha z k^{\alpha} l^{1-\alpha} = \lambda \alpha y$$

and

$$Wl = \lambda(1-\alpha) z k^{\alpha} l^{1-\alpha} = \lambda(1-\alpha) y$$

Adding these up we obtain

$$\mathsf{TC}(y) \equiv \min_{k,l} \Big[Rk + Wl \;\big|\; z k^{\alpha} l^{1-\alpha} = y \Big] = \lambda y$$

Hence marginal cost is a constant, $\mathsf{mc} = \mathsf{TC}'(y) = \lambda$, independent of the scale of production y. Marginal cost *is* the Lagrange multiplier λ because marginal cost is the increase in costs necessitated by a small increase in the scale of production. To solve for λ write

$$(Rk)^{\alpha} = (\lambda \alpha y)^{\alpha}$$

and

$$(Wl)^{1-\alpha} = (\lambda(1-\alpha) y)^{1-\alpha}$$

Multiplying these conditions together

$$(Rk)^{\alpha}(Wl)^{1-\alpha} = \lambda \alpha^{\alpha}(1-\alpha)^{1-\alpha} y$$

But since $z k^{\alpha} l^{1-\alpha} = y$ this is just

$$R^{\alpha} W^{1-\alpha} = \lambda \alpha^{\alpha}(1-\alpha)^{1-\alpha} z$$

So the Lagrange multiplier is

$$\lambda = \left(\frac{R}{\alpha}\right)^{\alpha} \left(\frac{W}{1-\alpha}\right)^{1-\alpha} \frac{1}{z}$$

So the cost function is indeed

$$\mathsf{TC}(y) = \lambda y = \left(\frac{R}{\alpha}\right)^{\alpha} \left(\frac{W}{1-\alpha}\right)^{1-\alpha} \frac{y}{z}$$

with marginal cost

$$\mathsf{mc} = \lambda = \left(\frac{R}{\alpha}\right)^{\alpha} \left(\frac{W}{1-\alpha}\right)^{1-\alpha} \frac{1}{z}$$

Markup pricing. Taking prices $p(i)$ as given, final good producers choose the bundle $y(i)$ to maximize

$$Y - \int_0^1 p(i) y(i)\, di$$

subject to

$$Y = \left(\int_0^1 y(i)^{1/\mu} \, di \right)^{\mu}$$

In other words they choose the bundle $y(i)$ to maximize

$$\left(\int_0^1 y(i)^{1/\mu} \, di \right)^{\mu} - \int_0^1 p(i)y(i) \, di$$

For each $i \in [0, 1]$ the first order condition can be written

$$\mu \left(\int_0^1 y(i)^{1/\mu} \, di \right)^{\mu-1} \frac{1}{\mu} y(i)^{\frac{1-\mu}{\mu}} - p(i) = 0$$

Simplifying and using the definition of Y this is the same as

$$Y^{\frac{\mu-1}{\mu}} y(i)^{\frac{1-\mu}{\mu}} = p(i)$$

This implies that the demand curve facing each intermediate producer is

$$y(i) = p(i)^{-\frac{\mu}{\mu-1}} Y$$

Each intermediate producer internalizes their market power and chooses $p(i)$ to maximize profits

$$\pi(i) \equiv \max_{p(i)} \left[(p(i) - \mathsf{mc})y(i) \,\middle|\, y(i) = p(i)^{-\frac{\mu}{\mu-1}} Y \right]$$

This profit maximization problem has the first order condition

$$\left[p(i)^{-\frac{\mu}{\mu-1}} - (p(i) - \mathsf{mc}) \frac{\mu}{\mu - 1} p(i)^{-\frac{\mu}{\mu-1}-1} \right] Y = 0$$

which solves for

$$p(i) = \mu \, \mathsf{mc}$$

Hence indeed each symmetric producer charges a price that is a markup $\mu > 1$ over marginal cost $\mathsf{mc} > 0$.

(b) **Representative household's problem.** Setting up the the individual household's Lagrangian

$$\mathcal{L} = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\log c_t - \frac{l_t^{1+\varphi}}{1+\varphi} \right) + \sum_{t=0}^{\infty} \lambda_t \left[W_t l_t + (R_t + 1 - \delta)k_t + \pi_t - c_t - k_{t+1} \right] \right\}$$

The key first order conditions for this problem can be written

$$c_t : \qquad \beta^t c_t^{-1} - \lambda_t = 0$$

$$l_t : \qquad -\beta^t l_t^{\varphi} + \lambda_t W_t = 0$$

$$k_{t+1} : \qquad -\lambda_t + \mathbb{E}_t \left\{ \lambda_{t+1}(R_{t+1} + 1 - \delta) \right\} = 0$$

Eliminating the multipliers in the usual way, we get the static labor supply condition equating the household's marginal rate of substitution to the real wage

$$c_t l_t^{\varphi} = W_t$$

and the consumption Euler equation

$$c_t^{-1} = \beta \mathbb{E}_t \left\{ c_{t+1}^{-1}(R_{t+1} + 1 - \delta) \right\}$$

(we also have the transversality condition and the initial condition for capital).

Representative firm's problem. In symmetric equilibrium we have $p_t(i) = p_t$ for all i. From the cost minimization conditions in part (a) above we have that marginal cost is the ratio of each factor's price to its physical marginal product

$$\mathsf{mc}_t = \frac{W_t}{(1-\alpha) z_t k_t^{\alpha} l_t^{-\alpha}} = \frac{R_t}{\alpha z_t k_t^{\alpha-1} l_t^{1-\alpha}}$$

and since $p_t = \mu \mathsf{mc}_t$ we can rewrite these as the factor demand conditions

$$W_t = \frac{p_t}{\mu}(1-\alpha) z_t k_t^{\alpha} l_t^{-\alpha} = (1-\alpha)\frac{p_t y_t}{l_t}$$

and

$$R_t = \frac{p_t}{\mu} \alpha z_t k_t^{\alpha-1} l_t^{1-\alpha} = \alpha \frac{p_t y_t}{k_t}$$

Aggregation and market clearing. Since $p_t(i) = p_t$ for all i we know from each intermediate producer's demand curve that each producer also has $y_t(i) = y_t$ for all i. Then from the final good production function

$$Y_t = \left(\int_0^1 y_t^{1/\mu}\, di \right)^{\mu} = y_t$$

which from the demand curve implies $y_t = p_t^{-\frac{\mu}{\mu-1}} y_t$ and hence $p_t = 1$ (this also implies that the perfectly competitive final goods producers make zero profits). With $p_t = 1$ we can then simplify the factor demands to

$$W_t = \frac{1-\alpha}{\mu} z_t k_t^{\alpha} l_t^{-\alpha} = \frac{1-\alpha}{\mu} \frac{y_t}{l_t}$$

and

$$R_t = \frac{\alpha}{\mu} z_t k_t^{\alpha-1} l_t^{1-\alpha} = \frac{\alpha}{\mu} \frac{y_t}{k_t}$$

We can then plug these expressions into the representative household's optimality conditions to get

$$c_t l_t^{\varphi} = \frac{1-\alpha}{\mu}\, z_t k_t^{\alpha} l_t^{-\alpha}$$

and

$$c_t^{-1} = \beta \mathbb{E}_t \left\{ c_{t+1}^{-1} \left(\frac{\alpha}{\mu}\, z_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta \right) \right\}$$

To derive the goods market clearing condition, first observe that intermediate profits are

$$\pi_t = (p_t - \mathsf{mc}_t)y_t = \left(p_t - \frac{p_t}{\mu}\right)y_t = \frac{\mu - 1}{\mu}y_t$$

with factor payments

$$W_t l_t = \frac{1-\alpha}{\mu}y_t$$
$$R_t k_t = \frac{\alpha}{\mu}y_t$$

Hence the total income of the representative household is

$$W_t l_t + R_t k_t + \pi_t = \frac{1-\alpha}{\mu}y_t + \frac{\alpha}{\mu}y_t + \frac{\mu-1}{\mu}y_t = y_t$$

So the goods market clearing condition is, as usual

$$c_t + k_{t+1} = y_t + (1-\delta)k_t = z_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t$$

Solving the model. In brief, to solve the model we first solve the following system of three equations

$$c_t l_t^{\varphi} = \frac{1-\alpha}{\mu} z_t k_t^{\alpha} l_t^{-\alpha}$$

$$c_t^{-1} = \beta \mathbb{E}_t \left\{ c_{t+1}^{-1} \left(\frac{\alpha}{\mu} z_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta \right) \right\}$$

and

$$c_t + k_{t+1} = z_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t$$

Given a stochastic process for z_t, these pin down the equilibrium c_t, l_t, k_t (and hence y_t) in the usual way. These equations coincide with the usual planning solution except for the μ terms in the factor demands. Given the solution for c_t, l_t, k_t and y_t we can then back out the W_t, R_t, π_t from the factor shares

$$W_t = \frac{1-\alpha}{\mu}\frac{y_t}{l_t}$$

$$R_t = \frac{\alpha}{\mu}\frac{y_t}{k_t}$$

$$\pi_t = \frac{\mu-1}{\mu}y_t$$

and of course we saw that $p_t = 1$ above. We have already seen that $Y_t = y_t$. The other aggregate quantities are simply given by $C_t = c_t L$, $K_t = k_t L$, $L_t = l_t L$. [Bad notation: Y_t here refers to final output per worker, while the other capital letters refer to true aggregates]

(c) **Steady state.** In a non-stochastic steady state with constant productivity level $\bar{z}$ we have from the consumption Euler equation

$$1 = \beta(\bar{R} + 1 - \delta) \qquad \Rightarrow \qquad \bar{R} = \rho + \delta, \qquad \rho \equiv \frac{1}{\beta} - 1$$

From the capital income share we then have the capital/output ratio

$$\frac{\bar{k}}{\bar{y}} = \frac{\alpha}{\mu}\frac{1}{\bar{R}} = \frac{\alpha}{\rho+\delta}\frac{1}{\mu}$$

From the production function we then have the capital/labor ratio

$$\frac{\bar{k}}{\bar{l}} = \left(\frac{\alpha}{\rho+\delta}\frac{\bar{z}}{\mu}\right)^{\frac{1}{1-\alpha}}$$

Hence the average product of labor is

$$\frac{\bar{y}}{\bar{l}} = \bar{z}^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\rho+\delta}\frac{1}{\mu}\right)^{\frac{\alpha}{1-\alpha}}$$

From the labor income share we then have the wage

$$\bar{W} = \frac{1-\alpha}{\mu}\frac{\bar{y}}{\bar{l}} = (1-\alpha)\left(\frac{\bar{z}}{\mu}\right)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

From the goods market clearing condition the consumption/output ratio is

$$\frac{\bar{c}}{\bar{y}} = 1-\delta\frac{\bar{k}}{\bar{y}} = \frac{(\rho+\delta)\mu-\alpha\delta}{(\rho+\delta)\mu}$$

To determine employment, we first write the labor market clearing condition as

$$\bar{l}^{\varphi}\bar{c} = \bar{W} = \frac{1-\alpha}{\mu}\frac{\bar{y}}{\bar{l}}$$

so

$$\bar{l}^{1+\varphi} = \frac{1-\alpha}{\mu}\frac{\bar{y}}{\bar{c}} = \frac{1-\alpha}{\mu}\left(\frac{(\rho+\delta)\mu}{(\rho+\delta)\mu-\alpha\delta}\right)$$

Hence steady state employment is

$$\bar{l} = \left(\frac{(1-\alpha)(\rho+\delta)}{(\rho+\delta)\mu-\alpha\delta}\right)^{\frac{1}{1+\varphi}}$$

We can then use the level of employment to recover steady state output $\bar{y}$ from

$$\bar{y} = \bar{z}^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\rho+\delta}\frac{1}{\mu}\right)^{\frac{\alpha}{1-\alpha}}\bar{l} = \bar{z}^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\rho+\delta}\frac{1}{\mu}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{(1-\alpha)(\rho+\delta)}{(\rho+\delta)\mu-\alpha\delta}\right)^{\frac{1}{1+\varphi}}$$

And similarly steady state capital $\bar{k}$ from

$$\bar{k} = \left(\frac{\alpha}{\rho+\delta}\frac{\bar{z}}{\mu}\right)^{\frac{1}{1-\alpha}}\bar{l} = \left(\frac{\alpha}{\rho+\delta}\frac{\bar{z}}{\mu}\right)^{\frac{1}{1-\alpha}}\left(\frac{(1-\alpha)(\rho+\delta)}{(\rho+\delta)\mu-\alpha\delta}\right)^{\frac{1}{1+\varphi}}$$

And steady state consumption $\bar{c}$ from

$$\bar{c} = \left(\frac{(\rho+\delta)\mu-\alpha\delta}{(\rho+\delta)\mu}\right)\bar{y}$$

And steady state profits $\bar{\pi}$ from

$$\bar{\pi} = \frac{\mu-1}{\mu}\bar{y}$$

(d) From the solutions in part (c) we see that the steady state rental rate $\bar{R}$ is independent of μ and so does not change. The capital/output ratio $\bar{k}/\bar{y}$ falls and hence the consumption/output ratio $\bar{c}/\bar{y}$ rises. The steady state capital/labor ration $\bar{k}/\bar{l}$ falls as does the steady state average product of labor $\bar{y}/\bar{l}$ and the wage rate $\bar{W}$. Steady state employment $\bar{l}$ falls. Since both $\bar{y}/\bar{l}$ and $\bar{l}$ fall, so does the level of output $\bar{y}$. Similarly since both $\bar{k}/\bar{y}$ and $\bar{y}$ fall, so does the level of capital $\bar{k}$. What about the level of consumption $\bar{c}$? We know that $\bar{c}/\bar{y}$ rises but $\bar{y}$ falls so these two effects move $\bar{c}$ in opposing directions. Intuitively, since $\bar{y}$ falls it must eventually be the case that $\bar{c}$ falls — but we can say more than this. In particular, to derive the net effect on consumption, observe that $\bar{c}$ satisfies the labor supply condition

$$\bar{l}^{\varphi}\bar{c} = \bar{W}$$

This is multiplicative and suggests an approach based on elasticities. In particular, taking logs and differentiating the effect of μ on $\bar{c}$ must satisfy

$$\varphi\frac{d\log\bar{l}}{d\log\mu} + \frac{d\log\bar{c}}{d\log\mu} = \frac{d\log\bar{W}}{d\log\mu}$$

Then use the solution for employment from part (c) above to calculate the semi-elasticity

$$\frac{d\log\bar{l}}{d\mu} = -\frac{1}{1+\varphi}\frac{(\rho+\delta)}{(\rho+\delta)\mu-\alpha\delta}$$

and since $d\log\mu = \frac{d\mu}{\mu}$ this implies the elasticity

$$\frac{d\log\bar{l}}{d\log\mu} = -\frac{1}{1+\varphi}\frac{(\rho+\delta)\mu}{(\rho+\delta)\mu-\alpha\delta}$$

We also have the elasticity of wages

$$\frac{d\log\bar{W}}{d\log\mu} = -\frac{1}{1-\alpha}$$

Combining these we see that the elasticity of consumption with respect to the markup is

$$\frac{d\log\bar{c}}{d\log\mu} = \frac{\varphi}{1+\varphi}\frac{(\rho+\delta)\mu}{(\rho+\delta)\mu-\alpha\delta} - \frac{1}{1-\alpha}$$

Rearranging, we see that

$$\frac{d\log\bar{c}}{d\log\mu} < 0 \qquad \Leftrightarrow \qquad \frac{\alpha\delta}{(1-\alpha)(\rho+\delta)} < \left(\frac{1}{1-\alpha} - \frac{\varphi}{1+\varphi}\right)\mu$$

Since $\alpha \in (0,1)$ and $\varphi > 0$ the term in brackets on the RHS of the inequality is positive and hence the RHS is strictly increasing in μ. Moreover since $\mu > 1$ it suffices to check the RHS at $\mu = 1$ since if the inequality is satisfied at $\mu = 1$ it is satisfied for all $\mu > 1$. Evaluating at $\mu = 1$ and rearranging we see that the sufficient condition is

$$\alpha\frac{\delta}{\rho+\delta} + (1-\alpha)\frac{\varphi}{1+\varphi} < 1$$

which is always satisfied — the LHS of this is a weighted average of $\frac{\delta}{\rho+\delta}$ and $\frac{\varphi}{1+\varphi}$, i.e., two numbers between 0 and 1 hence the LHS is also between 0 and 1. In short we see that the elasticity of $\bar{c}$ with respect to μ is always negative and hence steady state consumption always falls. So even though there are two offsetting effects of μ on $\bar{c}$, the net effect is unambiguously negative.

Finally steady state profits are given by $\bar{\pi} = \frac{\mu-1}{\mu}\bar{y}$. The profit rate $\bar{\pi}/\bar{y} = \frac{\mu-1}{\mu}$ is of course increasing in the markup μ. But output $\bar{y}$ is decreasing in μ so again there are two offsetting effects. Can we again see which dominates? Here it genuinely depends. For $\mu = 1$ we have $\bar{\pi} = 0$ but for any $\mu > 1$ we have $\bar{\pi} > 0$ so for μ close to 1 we have $\bar{\pi}$ increasing in μ (the profit rate effect dominates). But $\bar{y}$ is monotonically decreasing in μ while the profit rate is bounded above by 1 so as we make μ higher we can't make profits more than $\bar{y}$ and making μ asymptotically high drives $\bar{y}$ to zero. So for high enough μ we expect that profits are decreasing in μ (the level of output effect dominates). This suggests informally that $\bar{\pi}$ is single-peaked in μ, at first rising from $\bar{\pi} = 0$ at $\mu = 1$, peaking in the interior and then decreasing back to $\bar{\pi} = 0$ as $\mu \to \infty$. In short, while the profit share $\bar{\pi}/\bar{y}$ is monotonically increasing in the markup, the actual level of profits is at first increasing then decreasing in the markup.

(e) The attached Dynare file `ps2_question2.mod` solves the model with the given parameters and calculates the responses to both a 1% productivity shock and a 1% markup shock. In this version of the model the markup μ_t is time-varying and our key equations become

$$c_t l_t^{\varphi} = \frac{1-\alpha}{\mu_t} z_t k_t^{\alpha} l_t^{-\alpha}$$

$$c_t^{-1} = \beta \mathbb{E}_t \left\{ c_{t+1}^{-1} \left(\frac{\alpha}{\mu_{t+1}} z_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta \right) \right\}$$

$$c_t + k_{t+1} = z_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta) k_t$$

$$W_t = \frac{1-\alpha}{\mu_t} \frac{y_t}{l_t}$$

$$R_t = \frac{\alpha}{\mu_t} \frac{y_t}{k_t}$$

$$\pi_t = \frac{\mu_t - 1}{\mu_t} y_t$$

(there is a μ_{t+1} in the Euler equation because it enters the rental rate R_{t+1} that determines the return on capital).

The impulse response functions for the log-deviations of consumption, investment, output, employment, the wage rate, rental rate, and profits in response to a productivity shock are are shown in Figure 3 below. The impulse response functions for the log-deviations of consumption, investment, output, employment, the wage rate, rental rate, and profits in response to a markup shock are are shown in Figure 4 below. Note that in the labor market condition and the Euler equation productivity and the markup enter symmetrically but with opposite signs as z_t/μ_t but the markup shock does not enter the resource constraint. In this sense a markup shock acts like an adverse shock to labor demand and capital demand but does change the real resource constraint of the economy.

A 1% productivity shock increases consumption, investment, output, employment, wages, the rental rate, and profits on impact. On impact, employment rises hence output responds by more than 1-for-1 with productivity (by more than 1%). Consumption rises by less than

1-for-1 with output with the remainder invested so that physical capital builds up and output returns to steady state (slightly) more slowly than does productivity. On impact the rental rate of capital rises then falls (overshooting its long run level) as output falls back to steady state while capital continues to build up. Although markups do not respond to productivity, profits are higher because output is higher.

A 1% markup shock acts much like the mirror-image, decreasing investment, output, employment, wages, and the rental rate on impact. But profits rise. Although output is falling, the increase in the markup increases the profit rate $\pi_t/y_t = \frac{\mu_t - 1}{\mu_t}$ by enough that total profits rise. In other words, for these parameter values the profit rate effect discussed in part (d) dominates the level of output effect.

(f)

Standard deviations.

From Dynare the standard deviations for the three cases are

	c	k	i	y	l	w	r	pi	m	z
both	0.0409	0.0651	0.1706	0.0550	0.0257	0.0544	0.0512	0.1963	0.0320	0.0320
z only	0.0374	0.0535	0.1403	0.0494	0.0106	0.0425	0.0345	0.0494	0	0.0320
m only	0.0167	0.0370	0.0970	0.0242	0.0234	0.0339	0.0379	0.1900	0.0320	0

With both shocks, profits and investment are the most volatile. Consumption and employment are smoother than output. With the markup shocks turned off, profits are much less volatile (they move in proportion to output) and the volatility of all other variables is also lower. Notice that with markup shocks only, the model produces considerably smaller fluctuations in consumption, investment and output than with productivity shocks only. Given that the shocks are of equal size, this suggests the model produces more amplification in response to productivity shocks than in response to markup shocks. This is not uniformly true however, with markup shocks only the model produces almost as much volatility in employment as with both shocks together.

Correlations.

For both shocks together we have

	c	k	i	y	l	w	r	pi	m	z
c	1.0000	0.9608	0.5017	0.8846	0.2961	0.8922	-0.1154	-0.0053	-0.2330	0.7726
k	0.9608	1.0000	0.4039	0.8079	0.3507	0.8886	-0.1492	-0.2114	-0.4027	0.5824
i	0.5017	0.4039	1.0000	0.8472	0.8374	0.7733	0.7855	-0.3389	-0.5300	0.7666
y	0.8846	0.8079	0.8472	1.0000	0.6332	0.9649	0.3525	-0.1860	-0.4288	0.8878
l	0.2961	0.3507	0.8374	0.6332	1.0000	0.6955	0.8424	-0.7971	-0.8961	0.3353
w	0.8922	0.8886	0.7733	0.9649	0.6955	1.0000	0.3114	-0.3808	-0.5989	0.7397
r	-0.1154	-0.1492	0.7855	0.3525	0.8424	0.3114	1.0000	-0.5533	-0.5996	0.2508
pi	-0.0053	-0.2114	-0.3389	-0.1860	-0.7971	-0.3808	-0.5533	1.0000	0.9674	0.2489
m	-0.2330	-0.4027	-0.5300	-0.4288	-0.8961	-0.5989	-0.5996	0.9674	1.0000	0.0000
z	0.7726	0.5824	0.7666	0.8878	0.3353	0.7397	0.2508	0.2489	0.0000	1.0000

Consumption, investment, employment, wages, the rental rate, and productivity are procyclical (high in booms when output is high, low in recessions), profits and markups are countercyclical.

Without the markup shocks, profits are procyclical (since now the profit rate is constant, so profits only move with the level of output).

	c	k	i	y	l	w	r	pi	z
c	1.0000	0.9753	0.5960	0.9181	0.3800	0.9731	-0.1788	0.9181	0.8466
k	0.9753	1.0000	0.4039	0.8079	0.1663	0.8981	-0.3918	0.8079	0.7080
i	0.5960	0.4039	1.0000	0.8654	0.9692	0.7650	0.6834	0.8654	0.9320
y	0.9181	0.8079	0.8654	1.0000	0.7154	0.9847	0.2257	1.0000	0.9882
l	0.3800	0.1663	0.9692	0.7154	1.0000	0.5829	0.8421	0.7154	0.8141
w	0.9731	0.8981	0.7650	0.9847	0.5829	1.0000	0.0527	0.9847	0.9464
r	-0.1788	-0.3918	0.6834	0.2257	0.8421	0.0527	1.0000	0.2257	0.3723
pi	0.9181	0.8079	0.8654	1.0000	0.7154	0.9847	0.2257	1.0000	0.9882
z	0.8466	0.7080	0.9320	0.9882	0.8141	0.9464	0.3723	0.9882	1.0000

Without productivity shocks, profits are countercyclical. As markups rise, output and profits both fall together reflecting the fact that the output level effect dominates the profit rate effect, as shown in the impulse response functions in Figure 4 and discussed in part (e) above.

	c	k	i	y	l	w	r	pi	m
c	1.0000	0.9838	0.2333	0.7345	0.4113	0.7770	-0.0182	-0.5471	-0.5699
k	0.9838	1.0000	0.4039	0.8443	0.5681	0.8773	0.1614	-0.6883	-0.7080
i	0.2333	0.4039	1.0000	0.8313	0.9823	0.7934	0.9680	-0.9416	-0.9320
y	0.7345	0.8443	0.8313	1.0000	0.9207	0.9979	0.6652	-0.9699	-0.9762
l	0.4113	0.5681	0.9823	0.9207	1.0000	0.8934	0.9039	-0.9880	-0.9834
w	0.7770	0.8773	0.7934	0.9979	0.8934	1.0000	0.6152	-0.9520	-0.9601
r	-0.0182	0.1614	0.9680	0.6652	0.9039	0.6152	1.0000	-0.8270	-0.8112
pi	-0.5471	-0.6883	-0.9416	-0.9699	-0.9880	-0.9520	-0.8270	1.0000	0.9996
m	-0.5699	-0.7080	-0.9320	-0.9762	-0.9834	-0.9601	-0.8112	0.9996	1.0000

Figure 1: Comparison across steady state for different automation thresholds I

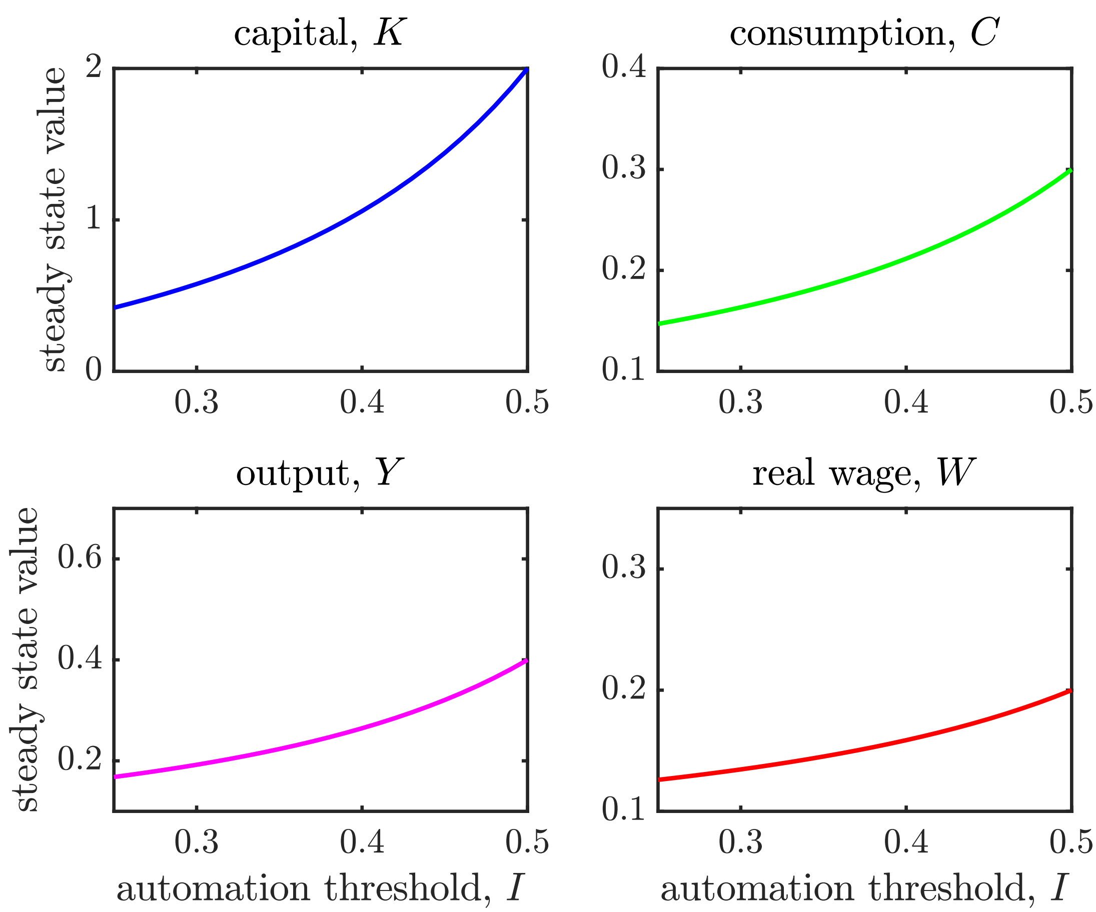

Figure 2: Comparison across steady state for different discount factors β

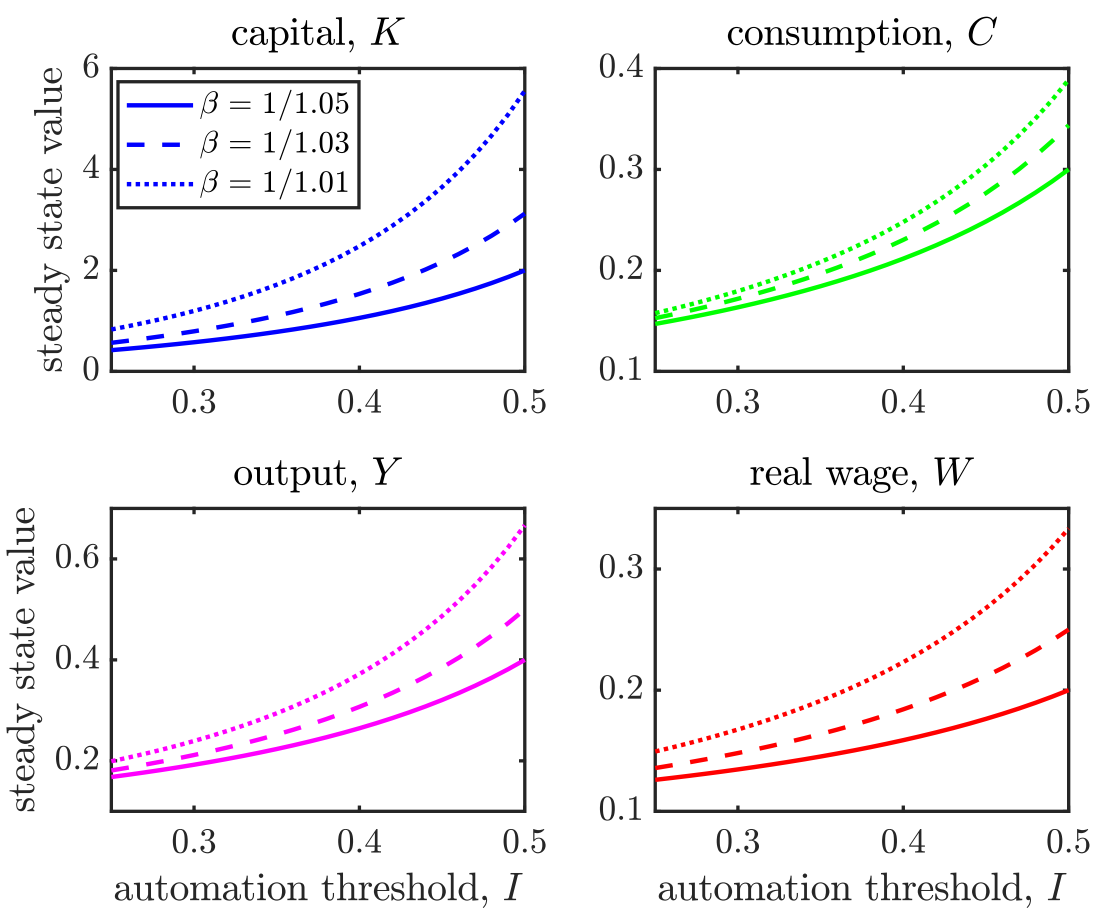

Figure 3: Response to 1% productivity shock

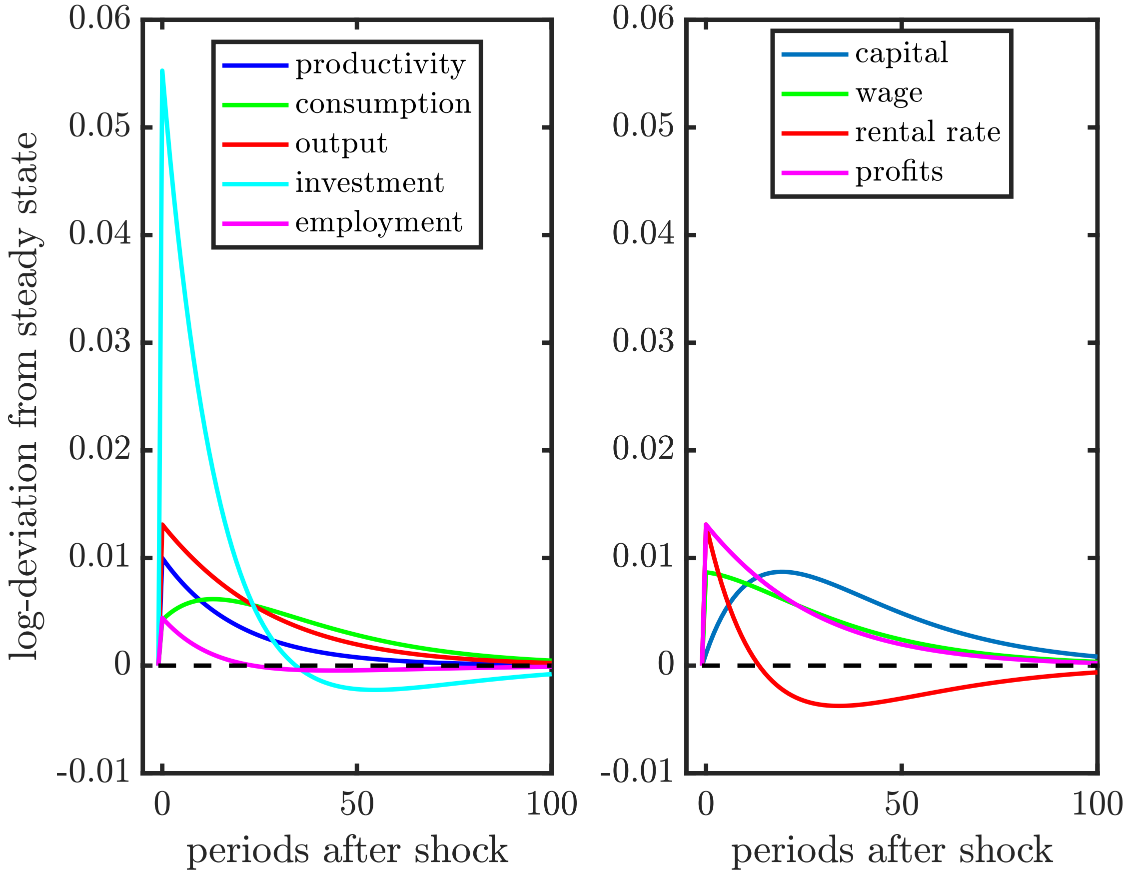

Figure 4: Response to 1% markup shock