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Advanced Macroeconomics
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Advanced Macroeconomics: Problem Set #1

Due Wednesday March 27 in class

1. **Solow model in continuous time.** Consider the Solow model in continuous time with production function $y = f(k)$ satisfying the usual properties, constant savings rate s, depreciation rate δ, productivity growth g and employment growth n.

   (a) Use the implicit function theorem to show how an increase in s affects the steady state values k^*, y^*, c^*. Does this change in s increase or decrease long run output and consumption per worker? Explain.

   Now consider the special case of a Cobb-Douglas production function $f(k) = k^\alpha$.

   (b) Derive the *elasticities* of steady state capital and output with respect to the savings rate

   $$\frac{d\log k^*}{d\log s}, \qquad \frac{d\log y^*}{d\log s}$$

   How do these depend on the curvature of the production function α? Explain.

   (c) Derive an exact solution for the time path $k(t)$ of capital per effective worker.

   Now consider the specific numerical values $\alpha = 0.3$, $s = 0.2$, $\delta = 0.05$, $g = 0.02$, $n = 0.03$.

   (d) Calculate and plot the time paths of $k(t), y(t), c(t)$ starting from the initial condition $k(0) = k^*/2$. How long is the *half-life* of convergence?

   (e) Now suppose that we are in steady state $k(0) = k^*$ when the savings rate suddenly increases to $s = 0.22$. Calculate and plot the time paths of $k(t), y(t), c(t)$ in response to this change. Explain both the short-run and long-run dynamics of $k(t), y(t), c(t)$. What if instead the savings rate had increased to $s = 0.30$? Do you think these are large or small effects on output? Explain.

2. **Natural resource depletion in the Solow model.** Consider a Solow model in continuous time where output is given by the production function

   $$Y(t) = K(t)^\alpha R(t)^\phi (A(t)L(t))^{1-\alpha-\phi}, \qquad 0 < \alpha, \phi < 1 \qquad \alpha + \phi < 1$$

   where $R(t)$ denotes a stock of *natural resources* that depletes at rate $\theta > 0$

   $$\dot{R}(t) = -\theta R(t)$$

   from some initial $R(0) > 0$. The rest of the model is as standard with constant savings rate s, depreciation rate δ, productivity growth g and employment growth n.

(a) Let $g_Y(t)$ and $g_K(t)$ denote the growth rates of output and the capital stock. Use the production function to derive a formula for $g_Y(t)$ in terms of $g_K(t)$.

(b) Let g_Y^* and g_K^* denote the growth rates of output and the capital stock along a balanced growth path. Show that along any balanced growth path $g_K^* = g_Y^*$. Solve for this growth rate in terms of model parameters.

(c) Does the economy necessarily converge to a balanced growth path? Explain.

(d) Now suppose instead that resources $R(t)$ grew in line with population, $\dot{R}(t) = nR(t)$. Compare the long-run growth rate of the economy with resource depletion from part (b) to the long growth rate of this alternative economy without resource depletion. What would make the gap between these growth rates large? Explain.

3. **Transitional dynamics in the Ramsey-Cass-Koopmans model.** Suppose the planner seeks to maximize the intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t \, u\Big(\frac{C_t}{L}\Big)L, \qquad 0 < \beta < 1$$

subject to the sequence of resource constraints

$$C_t + K_{t+1} = F(K_t, L) + (1-\delta)K_t, \qquad 0 < \delta < 1$$

given initial $K_0 > 0$. The production function has the Cobb-Douglas form

$$Y = F(K, L) = AK^\alpha L^{1-\alpha}, \qquad 0 < \alpha < 1$$

Suppose that productivity $A > 0$ and the labor force $L > 0$ are constant. Let $c_t = C_t/L$, $k_t = K_t/L$, $y_t = Y_t/L$ etc denote consumption, capital, output etc in *per worker* units. Suppose that the period utility function is strictly increasing and strictly concave.

(a) Derive optimality conditions that characterize the solution to the planner's problem. Give intuition for those optimality conditions. Explain how these optimality conditions pin down the dynamics of c_t and k_t.

(b) Solve for the steady state values c^*, k^*, y^* in terms of the parameters. How do these steady state values depend on the level of A?

(c) Suppose the economy is initially in the steady state you found in (b). Then suddenly there is a permanent increase in productivity from A to $A' > A$. Use a phase diagram to explain both the short-run and long-run dynamics of c_t and k_t in response to this increase in productivity. Does c_t increase or decrease? Explain.

Now consider the isoelastic utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \qquad \sigma > 0$$

(d) Log-linearize the planner's optimality conditions around the steady-state. Guess that in log-deviations capital satisfies

$$\hat{k}_{t+1} = \psi_{kk}\hat{k}_t$$

and that consumption satisfies

$$\hat{c}_t = \psi_{ck}\hat{k}_t$$

Use the method of undetermined coefficients to determine ψ_{kk} and ψ_{ck} in terms of model parameters. How if at all do these depend on the level of A?

Now consider the specific numerical values $\alpha = 0.3$, $\beta = 1/1.05$, $\delta = 0.05$, $A = 1$ and $\sigma = 1$.

(e) Calculate the values of ψ_{kk} and ψ_{ck}. Suppose the economy is at steady state when suddenly at $t = 0$ there is a 5% permanent increase in the level of productivity from $A = 1$ to $A' = 1.05$. Calculate the transitional dynamics of the economy as it adjusts to its new long run values. In particular, calculate and plot the time-paths of capital, output, and consumption until they have converged to their new steady state levels.

(f) How if at all would your answers to parts (d) and (e) change if σ was lower, say $\sigma = 0.5$? Or higher, say $\sigma = 2$? Give intuition for your answers.