Advanced Macroeconomics

Lecture 9: real business cycles, part one

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This class

- Stochastic growth model
  - exogenous shocks to productivity
  - outcomes are stochastic processes for consumption, capital etc
  - starting point for RBC-style models

Stochastic difference equations

- Consider scalar *stochastic difference equation*

$$x_{t+1} = (1-\phi)\bar{x} + \phi x_t + \varepsilon_{t+1}, \qquad x_0 \text{ given}$$

  A simple stochastic process. Also known as as a first-order autoregression or AR(1)

- Innovations ε_t drawn from given distribution, for example

$$\varepsilon_t \sim \text{ IID } N(0, \sigma_\varepsilon^2)$$

Stochastic difference equations

- Iterating forward from x_0 gives

$$x_t = \bar{x} + \phi^t (x_0 - \bar{x}) + \sum_{i=0}^{t-1} \phi^i \varepsilon_{t-i}$$

- The expected value of the moving average is

$$\mathbb{E} \left\{ \sum_{i=0}^{t-1} \phi^i \varepsilon_{t-i} \right\} = 0$$

while the variance is

$$\text{Var} \left\{ \sum_{i=0}^{t-1} \phi^i \varepsilon_{t-i} \right\} = \sigma_\varepsilon^2 \sum_{i=0}^{t-1} \phi^{2i}$$

(since the innovations are independent)

Limiting distribution

- Thus distribution at date t is

$$x_t \sim N\left(\bar{x} + \phi^t (x_0 - \bar{x}) \, , \, \sigma_\varepsilon^2 \sum_{i=0}^{t-1} \phi^{2i} \right)$$

- If $|\phi| < 1$, converges to well-behaved limiting distribution as $t \to \infty$

$$x \sim N\left(\bar{x} \, , \, \frac{\sigma_\varepsilon^2}{1 - \phi^2} \right)$$

(also known as the 'long-run' or 'stationary' distribution)

- Long-run variance depends on both innovation variance σ_ε^2 and persistence ϕ. A persistent process has higher long-run variance

- If $|\phi| \geq 1$, process is not stationary and does not converge to a limiting distribution

Simulating a simple AR(1) process

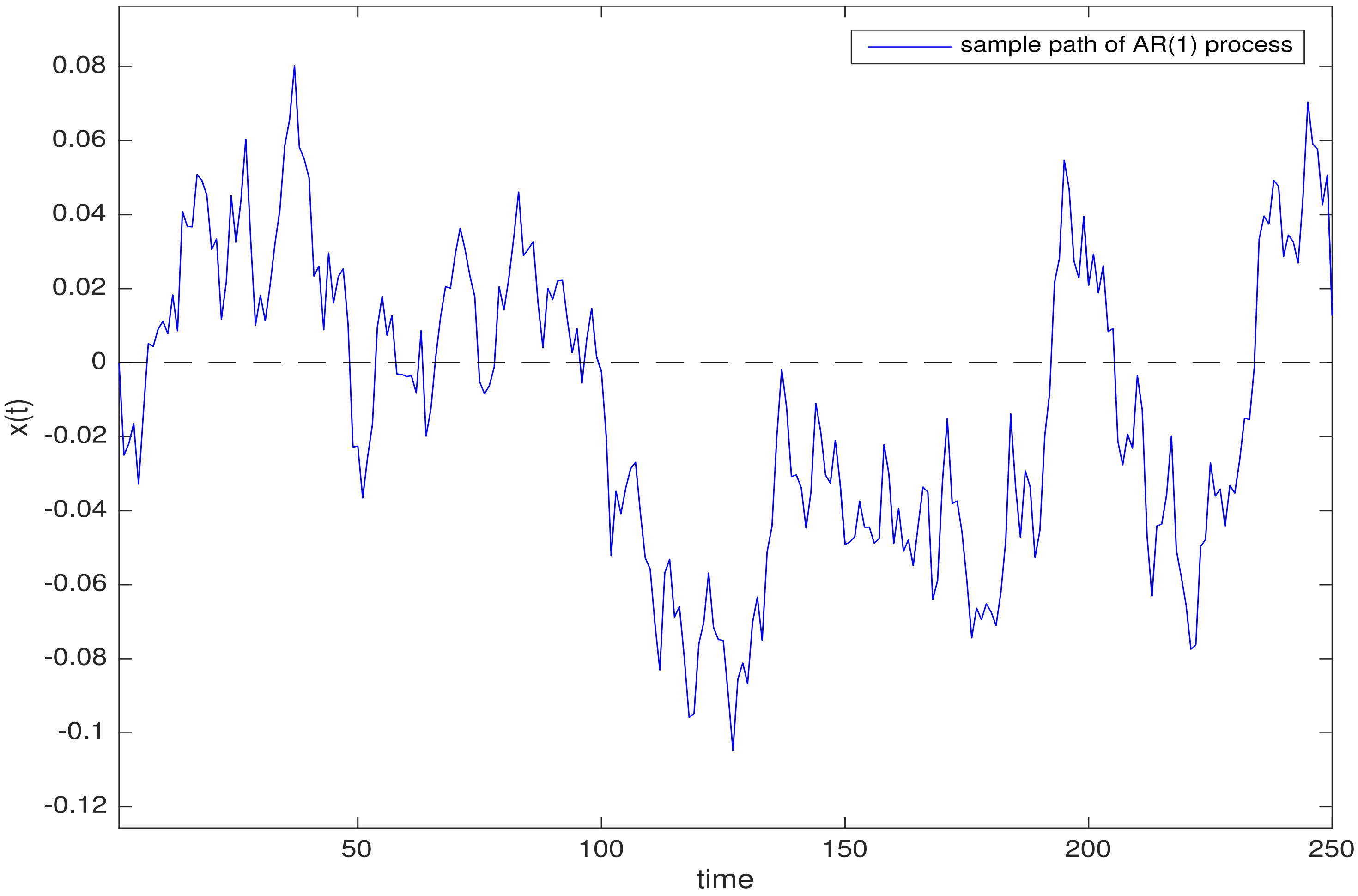
From Matlab script "*simulate_AR1_example.m*" in LMS

```
%%%%% parameters

S      = 250;   %% length of simulation

phi    = 0.5;  %% AR(1) coefficient
sigeps = 0.015; %% innovation standard deviation

x0     = 0;     %% initial condition

%%%%% draw S realizations from N(0,sigeps^2)

epsilons = sigeps*randn(S,1);
```

```
%%%%% iteratively construct sample path

xt     = zeros(S,1);

for s=2:S,

xt(1) = x0;
xt(s) = phi*xt(s-1)+epsilons(s);

end
```

```
time = (1:1:S)';

figure(1)
plot(time,xt,'b-',time,zeros(S,1),'k--')
xlabel('time')
ylabel('x(t)')
axis([min(time) max(time) 1.2*min(xt) 1.2*max(xt)])
legend('sample path of AR(1) process')
```

Simulated AR(1)



Stochastic growth model

- Social planner maximizes expected intertemporal utility

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}, \qquad 0 < \beta < 1$$

subject to sequence of resource constraints, for each date and state

$$c_t + k_{t+1} = z_t f(k_t) + (1 - \delta) k_t, \qquad 0 < \delta < 1$$

- Initial $k_0 > 0$ and stochastic process for productivity $\{z_t\}$ given

- All variables in per worker units. Simplified problem, abstracting from trend productivity growth and trend employment growth

Social planner's problem

- Lagrangian with stochastic multiplier $\lambda_t \geq 0$ for each constraint

$$\mathcal{L} = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \big[z_t f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \big] \right\}$$

- Some key first order conditions

$$c_t : \qquad \beta^t u'(c_t) - \lambda_t = 0$$

$$k_{t+1} : \qquad -\lambda_t + \mathbb{E}_t \left\{ \lambda_{t+1} \big[z_{t+1} f'(k_{t+1}) + 1 - \delta \big] \right\} = 0$$

$$\lambda_t : \qquad z_t f(k_t) + (1-\delta)k_t - c_t - k_{t+1} = 0$$

These hold at every date and state

- Although k_{t+1} has a $t+1$ subscript, it is chosen conditional on date t information

Dynamical system

- Gives a system of stochastic difference equations

$$u'(c_t) = \beta \mathbb{E}_t \left\{ u'(c_{t+1}) \left[z_{t+1} f'(k_{t+1}) + 1 - \delta \right] \right\}$$

and

$$c_t + k_{t+1} = z_t f(k_t) + (1 - \delta) k_t$$

given initial k_0 and transversality condition

- Maps exogenous stochastic process $\{z_t\}$ into endogenous stochastic processes $\{c_t, k_t\}$ etc

- In this sense, solution is a probability distribution for $\{c_t, k_t\}$ etc conditional on parameters of the model — i.e., a *likelihood function*

“Non-stochastic steady state”

- Shut down shocks, set $z_t = \bar{z}$
- Find steady state of associated deterministic model
- Steady state capital $\bar{k}$ solves

$$1 = \beta \big[\bar{z} f'(\bar{k}) + 1 - \delta \big]$$

- Steady state consumption $\bar{c}$ pinned down by resource constraint

$$\bar{c} = \bar{z} f(\bar{k}) - \delta \bar{k}$$

Log-linearization

- Resource constraint

$$c_t + k_{t+1} = z_t f(k_t) + (1-\delta)k_t$$

- Log-linear approximation

$$\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} \approx \bar{z}f(\bar{k})\hat{z}_t + \left[\bar{z}f'(\bar{k}) + 1 - \delta\right]\bar{k}\hat{k}_t$$

or

$$\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} \approx \bar{y}\hat{z}_t + \frac{1}{\beta}\bar{k}\hat{k}_t$$

Log-linearization

- Consumption Euler equation

$$u'(c_t) = \beta \mathbb{E}_t \left\{ u'(c_{t+1}) R_{t+1} \right\}, \quad R_{t+1} \equiv z_{t+1} f'(k_{t+1}) + 1 - \delta$$

- Log-linear approximation

$$u''(\bar{c})\bar{c}\hat{c}_t \approx \beta \mathbb{E}_t \left\{ \bar{R} u''(\bar{c})\bar{c}\hat{c}_{t+1} + u'(\bar{c})\bar{R}\hat{R}_{t+1} \right\}$$

so that on using $\beta \bar{R} = 1$ and defining $\sigma(c) \equiv -u''(c)c/u'(c)$ we have

$$\mathbb{E}_t \left\{ \Delta \hat{c}_{t+1} \right\} \approx \frac{1}{\sigma(\bar{c})} \mathbb{E}_t \left\{ \hat{R}_{t+1} \right\}$$

where similarly

$$\hat{R}_{t+1} \approx \beta \left[f'(\bar{k})\bar{z}\hat{z}_{t+1} + \bar{z} f''(\bar{k})\bar{k}\hat{k}_{t+1} \right]$$

Standard parameterization

- CRRA utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \qquad \sigma > 0$$

- Cobb-Douglas production function

$$f(k) = k^{\alpha}, \qquad 0 < \alpha < 1$$

- AR(1) process for log productivity, normalizing $\bar{z} = 1$

$$z_{t+1} = \phi z_t + \varepsilon_{t+1}, \qquad 0 < \phi < 1$$

with IID normal innovations

$$\varepsilon_{t+1} \sim N(0, \sigma_{\varepsilon}^2)$$

Standard parameterization

- Treating approximations as exact, with this parameterization the system of log-linear equations can be written

$$\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} = \bar{y}\hat{z}_t + \frac{1}{\beta}\bar{k}\hat{k}_t$$

and

$$\mathbb{E}_t\left\{\Delta\hat{c}_{t+1}\right\} = \frac{1}{\sigma}\mathbb{E}_t\left\{\beta\bar{r}[\hat{z}_{t+1} + (\alpha - 1)\hat{k}_{t+1}]\right\}$$

where $\bar{r}$ denotes the steady state marginal product of capital

$$\bar{r} \equiv f'(\bar{k}) = \alpha\bar{k}^{\alpha-1} = \rho + \delta$$

Method of undetermined coefficients

- Guess solutions linear in *state variables* $\hat{k}_t$ and $\hat{z}_t$

- For the endogenous state variable, capital

$$\hat{k}_{t+1} = \psi_{kk}\hat{k}_t + \psi_{kz}\hat{z}_t$$

- For the *control variable*, consumption

$$\hat{c}_t = \psi_{ck}\hat{k}_t + \psi_{cz}\hat{z}_t$$

- In short, we need to determine *four* coefficients

$$\psi_{kk},\ \psi_{ck},\ \psi_{kz},\ \psi_{cz}$$

Method of undetermined coefficients

- Note that

$$\hat{c}_{t+1} = \psi_{ck}\hat{k}_{t+1} + \psi_{cz}\hat{z}_{t+1}$$

$$= \psi_{ck}(\psi_{kk}\hat{k}_t + \psi_{kz}\hat{z}_t) + \psi_{cz}\hat{z}_{t+1}$$

- So expected consumption is

$$\mathbb{E}_t\{\hat{c}_{t+1}\} = \psi_{ck}\psi_{kk}\hat{k}_t + (\psi_{ck}\psi_{kz} + \psi_{cz}\phi)\hat{z}_t$$

(i.e., conditional expectations also linear in state variables)

Tedious algebra

- Resource constraint

$$\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} = \bar{y}\hat{z}_t + \frac{1}{\beta}\bar{k}\hat{k}_t$$

- Plug in guesses and collect terms

$$0 = \left[\bar{c}\psi_{ck} + \bar{k}\psi_{kk} - \frac{1}{\beta}\bar{k}\right]\hat{k}_t + \left[\bar{c}\psi_{cz} + \bar{k}\psi_{kz} - \bar{y}\right]\hat{z}_t$$

- This has to hold for *any* values of $\hat{k}_t, \hat{z}_t$. Gives two conditions

$$\boxed{\bar{c}\psi_{ck} + \bar{k}\psi_{kk} - \frac{1}{\beta}\bar{k} = 0} \tag{1}$$

and

$$\boxed{\bar{c}\psi_{cz} + \bar{k}\psi_{kz} - \bar{y} = 0} \tag{2}$$

Even more tedious algebra

- Consumption Euler equation as

$$\mathbb{E}_t\left\{\Delta\hat{c}_{t+1}\right\} = \frac{1}{\sigma}\mathbb{E}_t\left\{\beta\bar{r}(\hat{z}_{t+1} + (\alpha-1)\hat{k}_{t+1})\right\}$$

- Again, plug in guesses and collect terms

$$0 = \left[\psi_{ck}\psi_{kk} - \psi_{ck} - \frac{\beta\bar{r}}{\sigma}(\alpha-1)\psi_{kk}\right]\hat{k}_t$$
$$+\left[\psi_{ck}\psi_{kz} + \psi_{cz}\phi - \psi_{cz} - \frac{\beta\bar{r}}{\sigma}(\phi + (\alpha-1)\psi_{kz})\right]\hat{z}_t$$

- Gives two more conditions

$$\boxed{\psi_{ck}\psi_{kk} - \psi_{ck} - \frac{\beta\bar{r}}{\sigma}(\alpha-1)\psi_{kk} = 0} \tag{3}$$

and

$$\boxed{\psi_{ck}\psi_{kz} + \psi_{cz}\phi - \psi_{cz} - \frac{\beta\bar{r}}{\sigma}(\phi + (\alpha-1)\psi_{kz}) = 0} \tag{4}$$

- Four equations to solve for the four coefficients

Recursive structure

- Coefficients on response to capital ψ_{kk}, ψ_{ck} can be solved first

$$\psi_{ck} = \left(\frac{1}{\beta} - \psi_{kk}\right)\frac{\bar{k}}{\bar{c}} \tag{1}$$

$$\psi_{ck}\psi_{kk} - \psi_{ck} - \frac{\beta\bar{r}}{\sigma}(\alpha - 1)\psi_{kk} = 0 \tag{3}$$

- Plug (1) into (3) to get quadratic equation in ψ_{kk}

Familiar quadratic

- Familiar quadratic equation in ψ_{kk}

$$\psi_{kk}^2 - \left(1 - \frac{\beta\bar{r}(\alpha - 1))}{\sigma}\frac{\bar{c}}{\bar{k}} + \frac{1}{\beta}\right)\psi_{kk} + \frac{1}{\beta} = 0$$

- Same characteristic polynomial we had for the deterministic version

- Two roots, both positive, one stable and one unstable

- Let ψ_{kk} denote the stable root

Now recover other coefficients

- Plugging ψ_{kk} back into (1)

$$\psi_{ck} = \left(\frac{1}{\beta} - \psi_{kk}\right)\frac{\bar{k}}{\bar{c}}$$

- Using solution for ψ_{ck}, now solve (2) and (4) simultaneously for ψ_{kz}, ψ_{cz}. These equations are linear in ψ_{kz}, ψ_{cz} given ψ_{ck}

- Yet more tedious algebra gives

$$\psi_{kz} = \frac{\frac{\beta\bar{r}}{\sigma}\phi + (1-\phi)\frac{\bar{y}}{\bar{c}}}{\frac{\beta\bar{r}}{\sigma}(1-\alpha) + \psi_{ck} + (1-\phi)\frac{\bar{k}}{\bar{c}}}$$

and finally

$$\psi_{cz} = \frac{\bar{y}}{\bar{c}} - \psi_{kz}\frac{\bar{k}}{\bar{c}}$$

Example

- From Matlab script "*stochastic_growth_example.m*" in LMS

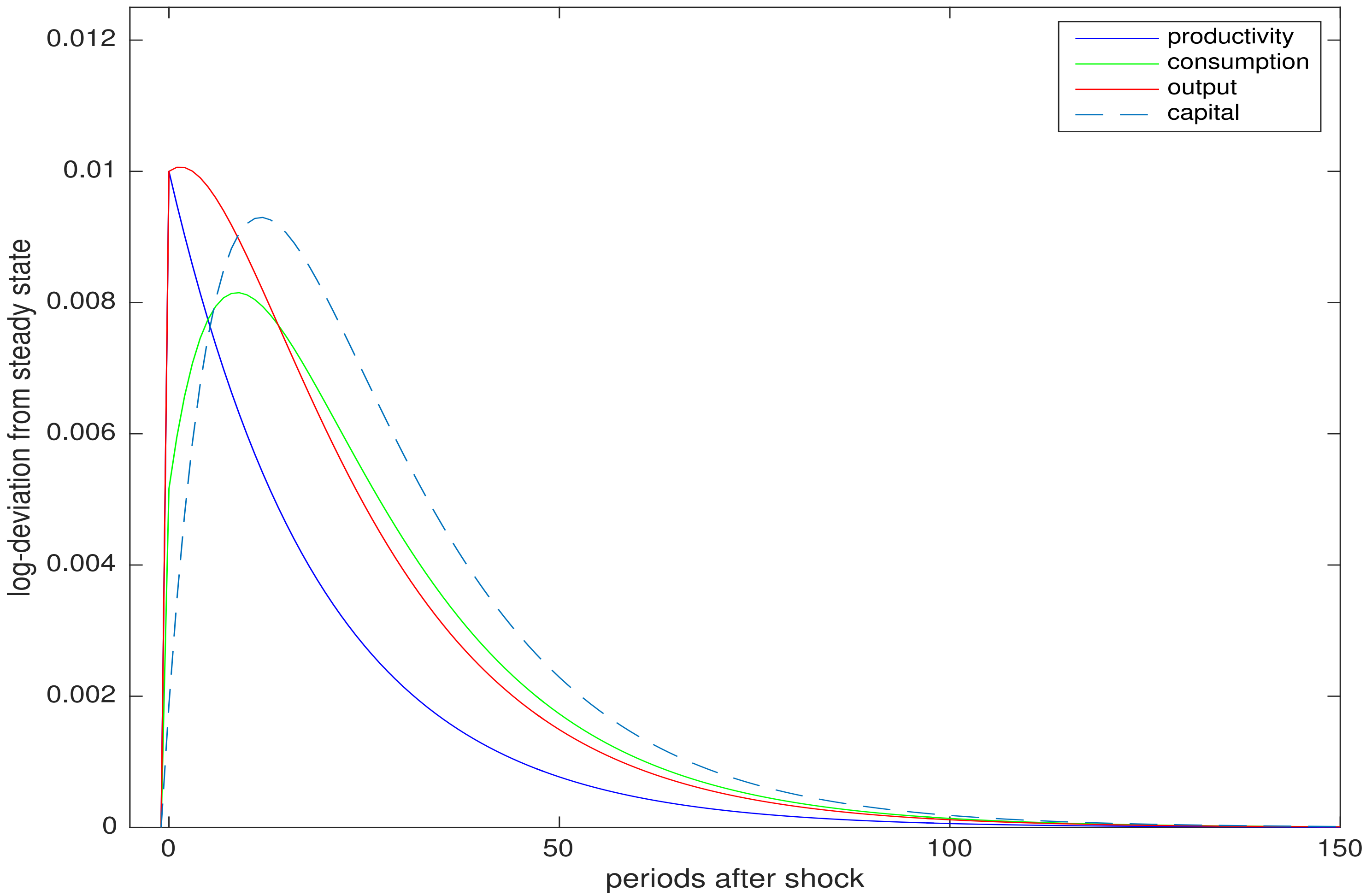
- Set parameters $\sigma = 1$ and $\alpha = 0.3$, $\beta = \phi = 0.95$ and $\delta = 0.05$

- Gives coefficients

$$\begin{pmatrix} \psi_{kk} & \psi_{kz} \\ \psi_{ck} & \psi_{cz} \end{pmatrix} = \begin{pmatrix} 0.89 & 0.19 \\ 0.56 & 0.52 \end{pmatrix}$$

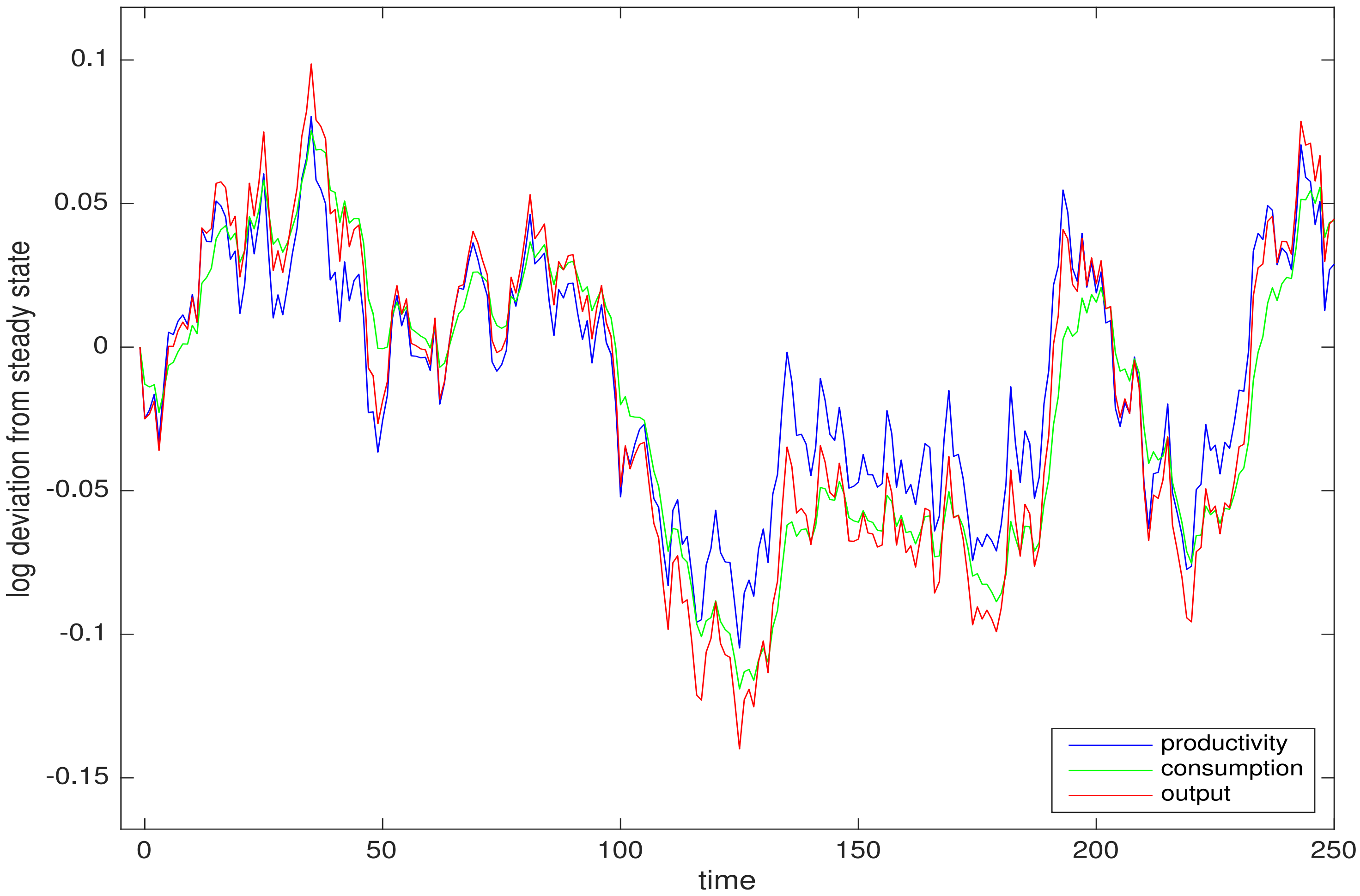
- Plots impulse response functions and a simulation

Impulse response functions



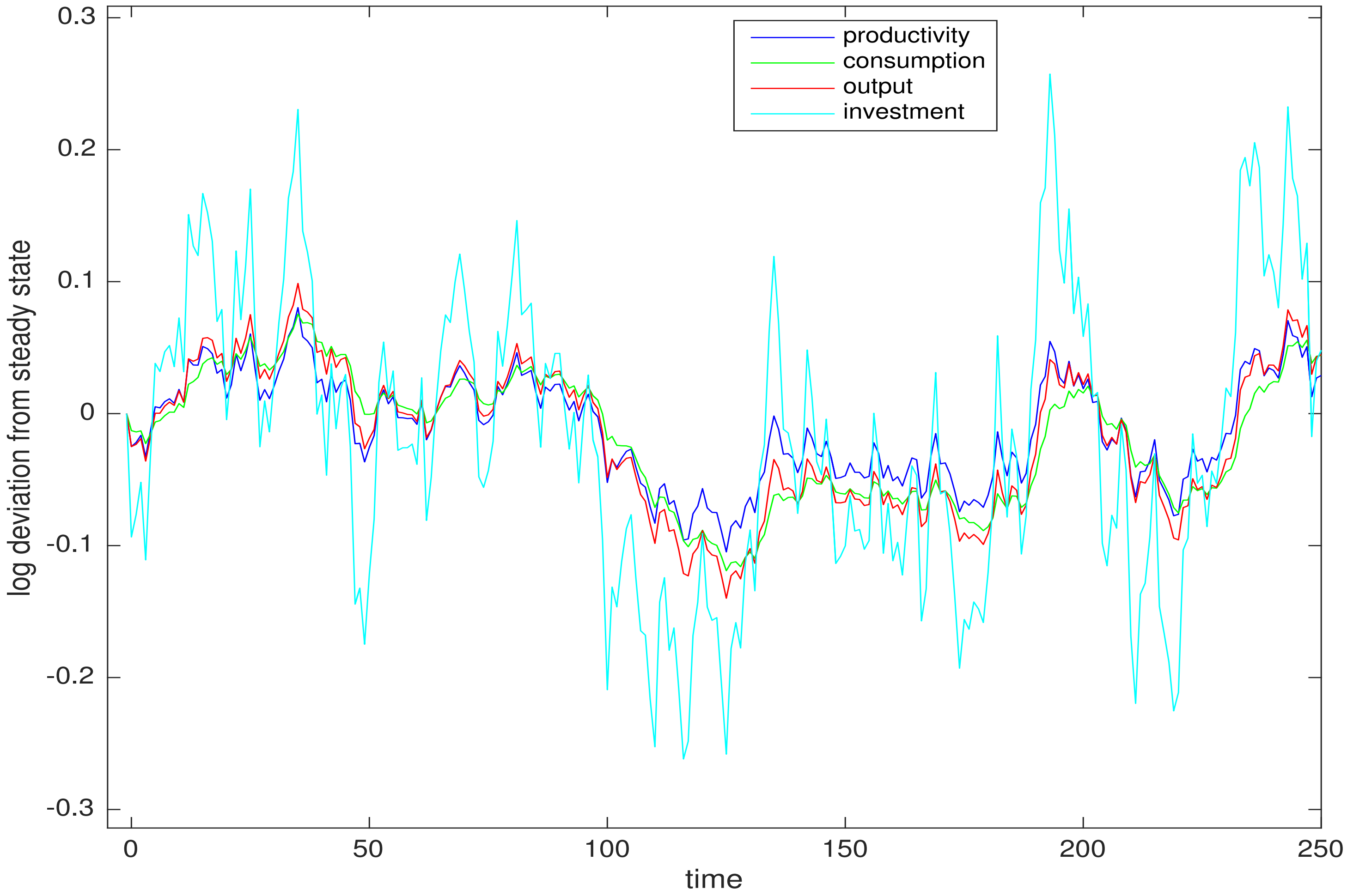
Innovation $\varepsilon_0 = 0.01$ (i.e., +1% productivity shock). Capital $\hat{k}_{t+1} = \psi_{kk}\hat{k}_t + \psi_{kz}\hat{z}_t$, consumption $\hat{c}_t = \psi_{ck}\hat{k}_t + \psi_{cz}\hat{z}_t$ and output $\hat{y}_t = \hat{z}_t + \alpha\hat{k}_t$ given $\hat{z}_t = \phi^t\varepsilon_0$.

Simulation



Capital $\hat{k}_{t+1} = \psi_{kk}\hat{k}_t + \psi_{kz}\hat{z}_t$, consumption $\hat{c}_t = \psi_{ck}\hat{k}_t + \psi_{cz}\hat{z}_t$, output $\hat{y}_t = \hat{z}_t + \alpha\hat{k}_t$ given simulated path for productivity $\hat{z}_t$.

Simulation



productivity
consumption
output
investment
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
log deviation from steady state
0
50
100
150
200
250
time

Next class

- Endogenous labor supply
  - proper RBC model with employment fluctuations
  - numerical examples, building intuition