Advanced Macroeconomics

Lecture 6: growth theory and dynamic optimization, part five

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This class

- Ramsey-Cass-Koopmans growth model in continuous time
  - brief introduction to optimal control theory
  - decentralization of planning problem

A standard optimal control problem

- Consider the problem of maximizing

$$\int_0^\infty e^{-\rho t}\, h(u(t), x(t))\, dt, \qquad \rho > 0$$

with '*state variable*' $x(t)$, '*control variable*' $u(t)$, and subject to the law of motion for the state

$$\dot{x}(t) = g(u(t), x(t)), \qquad x(0) = x_0 \text{ given}$$

and feasible controls $u(t) \in \mathcal{U}$

- Characterize solution of this problem using *Hamiltonian*

Hamiltonian

- Hamiltonian for this problem (in *current-value* representation)

$$\mathcal{H}(u, x, \lambda) \equiv h(u, x) + \lambda g(u, x)$$

- Key optimality conditions, for all $t \geq 0$,

$$\mathcal{H}_u(u(t), x(t), \lambda(t)) = 0$$

$$\mathcal{H}_x(u(t), x(t), \lambda(t)) = \rho\lambda(t) - \dot{\lambda}(t)$$

$$\mathcal{H}_\lambda(u(t), x(t), \lambda(t)) = \dot{x}(t)$$

Along with initial condition $x(0) = x_0$ and transversality condition

$$\lim_{T\to\infty} e^{-\rho T}\lambda(T)x(T) = 0$$

Ramsey-Cass-Koopmans

- State variable capital k, control variable consumption c

- Planner's problem is to maximize

$$\int_0^\infty e^{-\rho t}\, u(c(t))\, dt, \qquad \rho > 0$$

subject to resource constraint

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t), \qquad k(0) = k_0 \text{ given}$$

and feasible consumption $c(t) \geq 0$

- Hamiltonian for this problem

$$\mathcal{H}(c, k, \lambda) \equiv u(c) + \lambda(f(k) - \delta k - c)$$

- Note simplified setup: constant employment and productivity

$$\mathcal{H}(c,k,\lambda) \equiv u(c) + \lambda(f(k) - \delta k - c)$$

- Key optimality conditions, for all $t \geq 0$,

$$\mathcal{H}_c(c(t),k(t),\lambda(t)) = 0$$
$$\mathcal{H}_k(c(t),k(t),\lambda(t)) = \rho\lambda(t) - \dot{\lambda}(t)$$
$$\mathcal{H}_\lambda(c(t),k(t),\lambda(t)) = \dot{k}(t)$$

along with initial condition and transversality condition

- Calculating the derivatives of the Hamiltonian

$$\mathcal{H}_c(c,k,\lambda) = u'(c) - \lambda$$
$$\mathcal{H}_k(c,k,\lambda) = \lambda(f'(k) - \delta)$$
$$\mathcal{H}_\lambda(c,k,\lambda) = f(k) - \delta k - c$$

Ramsey-Cass-Koopmans

- Hence system of optimality conditions can be written

$$u'(c(t)) = \lambda(t)$$
$$\dot{\lambda}(t) = (\rho - (f'(k(t)) - \delta))\lambda(t)$$
$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$$

- Differentiating the first condition with respect to t gives

$$u''(c(t))\dot{c}(t) = \dot{\lambda}(t)$$

- So the first two conditions can be combined to eliminate $\lambda(t)$, giving the continuous time consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\sigma(c(t))}, \qquad \sigma(c) \equiv -\frac{u''(c)c}{u'(c)}$$

where $\sigma(c)$ is the Arrow/Pratt coefficient of relative risk aversion

Ramsey-Cass-Koopmans

- System of differential equations in $c(t), k(t)$

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\sigma(c(t))}$$

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$$

given initial condition $k(0) = k_0$ and transversality condition

$$\lim_{T \to \infty} e^{-\rho T} u'(c(T))k(T) = 0$$

- One given initial condition $k(0)$, initial consumption $c(0)$ can jump

- Unique solution if dynamical system has one stable and one unstable root

Continuous vs. discrete time

- Suppose isoelastic utility

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \qquad \sigma > 0$$

- Continuous time consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\sigma}$$

- Discrete time consumption Euler equation

$$\frac{c_{t+1}}{c_t} = \left(\beta[f'(k_{t+1}) + 1 - \delta]\right)^{1/\sigma}$$

so that on taking logs and using $\log(1 + x) \approx x$ we have

$$\Delta \log c_{t+1} \approx \frac{f'(k_{t+1}) - \delta - \rho}{\sigma}$$

Steady state

- Steady state c^*, k^* where $\dot{c}(t) = 0$ and $\dot{k}(t) = 0$, implied by

  $$f'(k^*) = \rho + \delta$$

  and

  $$c^* = f(k^*) - \delta k^*$$

- As usual c^*, k^* independent of $u(c)$ function

Qualitative dynamics

- From consumption Euler equation

$$\dot{c}(t) > 0 \quad \Leftrightarrow \quad f'(k(t)) > \rho + \delta \quad \Leftrightarrow \quad k(t) < k^*$$

- Let $C(k)$ denote consumption sustained by holding $k(t)$ fixed at k

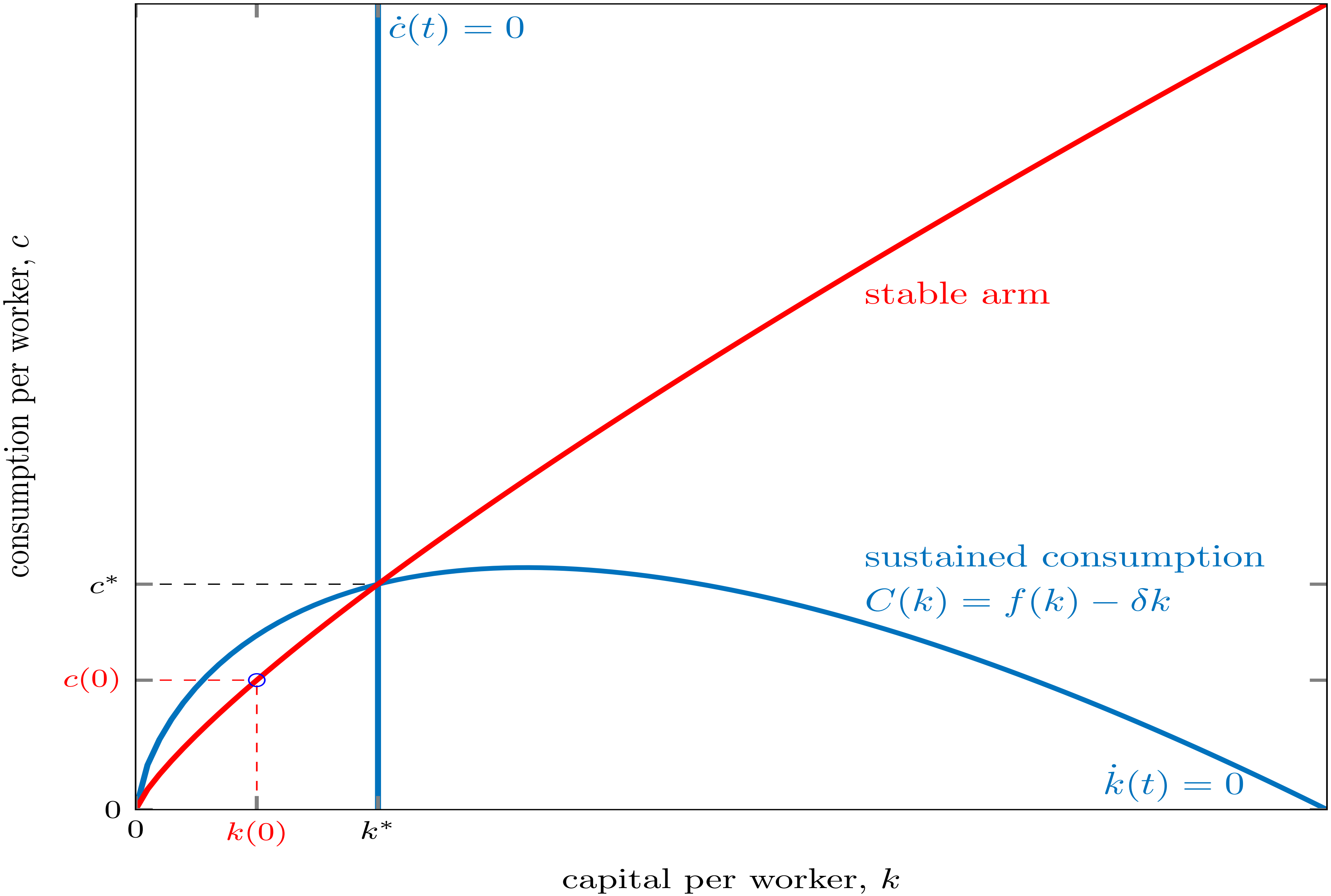
$$C(k) \equiv f(k) - \delta k$$

- Then from resource constraint

$$\dot{k}(t) > 0 \quad \Leftrightarrow \quad f(k(t)) - \delta k(t) > c(t) \quad \Leftrightarrow \quad C(k(t)) > c(t)$$

- Analyze these qualitative dynamics in a phase diagram

Phase diagram in $k(t), c(t)$ space



Linear differential equations

- Consider scalar linear differential equation

$$\dot{x}(t) = ax(t) + b, \qquad x(0) = x_0 \text{ given}$$

- Steady state, if $a \neq 0$

$$\bar{x} = -a^{-1}b$$

- Solution, if $a \neq 0$

$$x(t) = \bar{x} + e^{at}(x(0) - \bar{x}), \qquad t \geq 0$$

If $a < 0$ then $x(t)$ *converges* (monotonically) to x^* as $t \to \infty$. If $a > 0$ then $x(t)$ *diverges* to $\pm\infty$ depending on sign of $x(0) - \bar{x}$

System of linear differential equations

- Now let's consider a *system* of linear differential equations

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

or in matrix notation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}$$

- Analogous steady state

$$\bar{\boldsymbol{x}} = -\boldsymbol{A}^{-1}\boldsymbol{b}$$

so that

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}})$$

System of linear differential equations

- Suppose $\boldsymbol{A}$ can be diagonalized

$$\boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V} = \boldsymbol{\Lambda}$$

- Then make change of variables $\boldsymbol{z}(t) = \boldsymbol{V}^{-1}(\boldsymbol{x}(t) - \bar{\boldsymbol{x}})$ and study the uncoupled system

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{\Lambda}\boldsymbol{z}(t)$$

- Solving the uncoupled system

$$\boldsymbol{z}(t) = e^{\boldsymbol{\Lambda}t}\boldsymbol{z}(0)$$

where the matrix exponential $e^{\boldsymbol{\Lambda}t}$ is simply a diagonal matrix with entries of the form $e^{\lambda t}$. In original coordinates

$$\boldsymbol{x}(t) = \bar{\boldsymbol{x}} + \boldsymbol{V}\boldsymbol{z}(t) = \bar{\boldsymbol{x}} + \boldsymbol{V}e^{\boldsymbol{\Lambda}t}\boldsymbol{z}(0)$$

System of linear differential equations

- That is, linear combinations of the form

$$x_1(t) = \bar{x}_1 + v_{11}e^{\lambda_1 t}z_1(0) + v_{12}e^{\lambda_2 t}z_2(0)$$

$$x_2(t) = \bar{x}_2 + v_{21}e^{\lambda_1 t}z_1(0) + v_{22}e^{\lambda_2 t}z_2(0)$$

- Stable roots $\lambda < 0$, unstable roots $\lambda > 0$. Note initial conditions

$$z_1(0) = \frac{v_{22}(x_1(0) - \bar{x}_1) - v_{12}(x_2(0) - \bar{x}_2)}{v_{11}v_{22} - v_{12}v_{21}}$$

$$z_2(0) = \frac{v_{11}(x_2(0) - \bar{x}_2) - v_{21}(x_1(0) - \bar{x}_1)}{v_{11}v_{22} - v_{12}v_{21}}$$

- An unstable λ dominates unless initial conditions 'just right'

Saddle path dynamics

- Suppose saddle path dynamics with

$$\lambda_1 < 0 < \lambda_2$$

- Then system explodes unless

$$z_2(0) = 0 \qquad \Leftrightarrow \qquad x_2(0) = \bar{x}_2 + \frac{v_{21}}{v_{11}}(x_1(0) - \bar{x}_1)$$

If system starts on this line ('*stable arm*','*stable manifold*') then converges to steady state. Diverges for any other initial conditions

Ramsey-Cass-Koopmans

- Nonlinear system of the form

$$\begin{pmatrix} \dot{c}(t) \\ \dot{k}(t) \end{pmatrix} = \begin{pmatrix} g_1(c(t), k(t)) \\ g_2(c(t), k(t)) \end{pmatrix}$$

where, for the usual isoelastic case

$$g_1(c, k) \equiv \frac{f'(k) - \rho - \delta}{\sigma} c, \qquad g_2(c, k) \equiv f(k) - \delta k - c$$

- Approximate dynamics

$$\begin{pmatrix} \dot{c}(t) \\ \dot{k}(t) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial c} g_1(c, k) & \frac{\partial}{\partial k} g_1(c, k) \\ \frac{\partial}{\partial c} g_2(c, k) & \frac{\partial}{\partial k} g_2(c, k) \end{pmatrix} \begin{pmatrix} c(t) - \bar{c} \\ k(t) - \bar{k} \end{pmatrix}$$

where the Jacobian matrix is evaluated at steady state $\bar{c}, \bar{k}$

- Local stability depends on signs of eigenvalues of this Jacobian

Ramsey-Cass-Koopmans

- Elements of the Jacobian matrix, evaluated at steady state

$$\frac{\partial}{\partial c} g_1(c,k) = \frac{f'(k) - \rho - \delta}{\sigma} = 0 \qquad \text{at } k = \bar{k}$$

$$\frac{\partial}{\partial k} g_1(c,k) = \frac{f''(k)}{\sigma} c < 0$$

$$\frac{\partial}{\partial c} g_2(c,k) = -1$$

$$\frac{\partial}{\partial k} g_2(c,k) = f'(k) - \delta = \rho \qquad \text{at } k = \bar{k}$$

Ramsey-Cass-Koopmans

- Let $\boldsymbol{A}$ denote this Jacobian matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & \frac{f''(\bar{k})}{\sigma}\bar{c} \\ -1 & \rho \end{pmatrix}$$

- Eigenvalues characterized by determinant

$$\det(\boldsymbol{A}) = \lambda_1 \lambda_2 = \frac{f''(\bar{k})}{\sigma}\bar{c} < 0$$

and trace

$$\mathrm{tr}(\boldsymbol{A}) = \lambda_1 + \lambda_2 = \rho > 0$$

- Hence roots real and of either sign, say

$$\lambda_1 < 0 < \lambda_2$$

and hence, as anticipated, exhibits saddle path dynamics

Compute the eigenvalues

- Characteristic polynomial

$$p(\lambda) = \lambda^2 - \text{tr}(\boldsymbol{A}) + \det(\boldsymbol{A})$$

- Solve the quadratic

$$p(\lambda) = \lambda^2 - \rho\lambda + \frac{f''(\bar{k})}{\sigma}\bar{c} = 0$$

gives roots

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4\frac{f''(\bar{k})\bar{c}}{\sigma}}}{2} < 0 < \frac{\rho + \sqrt{\rho^2 - 4\frac{f''(\bar{k})\bar{c}}{\sigma}}}{2} = \lambda_2$$

Method of undetermined coefficients

- Write out approximate dynamics

$$\dot{c}(t) = \frac{f''(\bar{k})\bar{c}}{\sigma}(k(t) - \bar{k})$$

and

$$\dot{k}(t) = -(c(t) - \bar{c}) + \rho(k(t) - \bar{k})$$

- Write this as a *second-order differential* equation in $k(t)$, namely

$$\ddot{k}(t) = \rho\dot{k}(t) - \frac{f''(\bar{k})\bar{c}}{\sigma}(k(t) - \bar{k})$$

- Now guess linear law of motion

$$\dot{k}(t) = \lambda(k(t) - \bar{k})$$

which implies that also

$$\ddot{k}(t) = \lambda\dot{k}(t) = \lambda^2(k(t) - \bar{k})$$

Method of undetermined coefficients

- Plug in guesses and collect terms

$$\left[\lambda^2 - \rho\lambda + \frac{f''(\bar{k})\bar{c}}{\sigma}\right](k(t) - \bar{k}) = 0$$

- Has to hold for *any* value of $(k(t) - \bar{k})$, gives us again

$$\lambda^2 - \rho\lambda + \frac{f''(\bar{k})\bar{c}}{\sigma} = 0$$

which implies the roots given on slide 21 above

- Also implies slope of the stable arm

$$c(t) - \bar{c} = (\rho - \lambda)(k(t) - \bar{k})$$

where $\lambda < 0$ denotes the stable root. Hence stable arm steeper than $\dot{k}(t) = 0$ locus

Decentralized problem: households

- Endowed with initial capital stock $k(0) > 0$, depreciation rate δ

- Endowed with one unit of labor, $l = 1$

- Supply $k(t)$ and $l = 1$ to competitive firms for $R(t)$ and $w(t)$

- Net assets $a(t)$ return $r(t)$

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t)$$

- Physical capital and other assets perfect substitutes (no risk), so no arbitrage implies

$$R(t) = r(t) + \delta$$

Decentralized problem: households

- Household problem is to choose $c(t) \geq 0$ to maximize

$$U = \int_0^\infty e^{-\rho t}\, u(c(t))\, dt$$

subject to the flow budget constraint

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t)$$

- A '*no-Ponzi-game*' constraint rules out large negative $a(t)$

$$\lim_{T\to\infty} q(T)a(T) \geq 0, \qquad q(t) \equiv \exp\big(-\int_0^t r(s)\, ds\big)$$

where $q(t)$ is the intertemporal price of consumption

$$\mathcal{H}(c, a, \lambda) \equiv u(c) + \lambda(ra + w - c)$$

- Key optimality conditions, for all $t \geq 0$,

$$\mathcal{H}_c(c(t), a(t), \lambda(t)) = 0$$
$$\mathcal{H}_a(c(t), a(t), \lambda(t)) = \rho\lambda(t) - \dot{\lambda}(t)$$
$$\mathcal{H}_\lambda(c(t), a(t), \lambda(t)) = \dot{a}(t)$$

along with initial condition and no-Ponzi condition etc

- Calculating the derivatives of the Hamiltonian

$$\mathcal{H}_c(c, a, \lambda) = u'(c) - \lambda$$
$$\mathcal{H}_a(c, a, \lambda) = \lambda r$$
$$\mathcal{H}_\lambda(c, a, \lambda) = ra + w - c$$

Decentralized problem: households

- Hence system of optimality conditions can be written

$$
\begin{aligned}
u'(c(t)) &= \lambda(t) \\
\dot{\lambda}(t) &= (\rho - r(t))\lambda(t) \\
\dot{a}(t) &= r(t)a(t) + w(t) - c(t)
\end{aligned}
$$

- Differentiating the first condition with respect to t gives

$$u''(c(t))\dot{c}(t) = \dot{\lambda}(t)$$

- If $u(c)$ is isoelastic, we have the simple consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\sigma}$$

Hence consumption is growing if $r(t) > \rho$ with interest sensitivity $\frac{1}{\sigma}$

Decentralized problem: firms

- Hire capital K and labor L to maximize profits

$$F(K, L) - RK - wL$$

- First order conditions

$$F_K(K, L) = R$$

$$F_L(K, L) = w$$

- In per worker terms and using no arbitrage condition $R = r + \delta$

$$f'(k) = r + \delta$$

$$f(k) - f'(k)k = w$$

Decentralized problem: equilibrium

- **Equilibrium**: (i) households maximize utility taking prices as given, (ii) firms maximize profits taking prices as given, and (iii) markets clear

$$L = 1, \qquad \text{and} \qquad k = a$$

- Implies system of differential equations

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\sigma} = \frac{f'(k(t)) - \delta - \rho}{\sigma}$$

and

$$\begin{aligned}
\dot{k}(t) &= \dot{a}(t) \\
&= r(t)a(t) + w(t) - c(t) \\
&= [f'(k(t)) - \delta]k(t) + [f(k(t)) - f'(k(t))k(t)] - c(t) \\
&= f(k(t)) - \delta k(t) - c(t)
\end{aligned}$$

- Coincides with planning problem

Alternative approach to household problem

- Integrate up the flow budget constraints to get consolidated *intertemporal budget constraint*

$$\int_0^\infty q(t)c(t)\,dt = a(0) + \int_0^\infty q(t)w(t)\,dt$$

in terms of the intertemporal prices $q(t)$

- Then form the Lagrangian

$$\mathcal{L} = \int_0^\infty e^{-\rho t}u(c(t))\,dt + \lambda\left(a(0) + \int_0^\infty q(t)[w(t) - c(t)]\,dt\right)$$

with *single* (constant) multiplier λ

Alternative approach to household problem

- First order condition for $c(t)$ is then just

$$e^{-\rho t}u'(c(t)) = \lambda q(t)$$

- Differentiating with respect to t gives

$$-\rho e^{-\rho t}u'(c(t)) + e^{-\rho t}u''(c(t))\dot{c}(t) = \lambda \dot{q}(t)$$

- Then note

$$\dot{q}(t) = -r(t)q(t)$$

- If $u(c)$ is isoelastic, again have simple consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\sigma}$$

Next class

- Some further topics in growth theory
  - technological change
  - capital-labor substitution vs. automation
  - imperfect competition