Advanced Macroeconomics

Lecture 5: growth theory
and dynamic optimization, part four

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This class

- Analyzing the dynamics of the Ramsey-Cass-Koopmans model
  - a system of nonlinear difference equations
  - log-linearization (convenient local approximation)
  - solving model by method of undetermined coefficients
  - examples and introduction to Matlab

Log-linearization

- Consider scalar function

$$y_t = f(x_t)$$

with steady state satisfying $\bar{y} = f(\bar{x})$. Local deviation in levels

$$y_t - \bar{y} \approx f'(\bar{x})(x_t - \bar{x})$$

or

$$\frac{y_t - \bar{y}}{\bar{y}} \approx \left(\frac{f'(\bar{x})\bar{x}}{f(\bar{x})}\right) \frac{x_t - \bar{x}}{\bar{x}}$$

- Note the following approximation for log-deviations

$$\hat{y}_t \equiv \log\left(\frac{y_t}{\bar{y}}\right) = \log\left(1 + \frac{y_t - \bar{y}}{\bar{y}}\right) \approx \frac{y_t - \bar{y}}{\bar{y}}$$

Log-linearization

- Hence in log-deviations coefficients are elasticities (units free)

$$\hat{y}_t \approx \left(\frac{f'(\bar{x})\bar{x}}{f(\bar{x})} \right) \hat{x}_t$$

- Generalizes naturally to multivariate functions, if $y_t = f(x_t, z_t)$ with $\bar{y} = f(\bar{x}, \bar{z})$ then

$$\hat{y}_t \approx \frac{f_x(\bar{x}, \bar{z})\bar{x}}{f(\bar{x}, \bar{z})} \hat{x}_t + \frac{f_z(\bar{x}, \bar{z})\bar{z}}{f(\bar{x}, \bar{z})} \hat{z}_t$$

where f_x and f_z denote partial derivatives

Examples

- Power functions

$$y = x^a z^b$$

implies

$$\hat{y} = a\hat{x} + b\hat{z}$$

(of course, since exactly log-linear)

- Linear functions

$$y = ax + bz$$

implies

$$\hat{y} = \left(\frac{a\bar{x}}{a\bar{x} + b\bar{z}}\right)\hat{x} + \left(\frac{b\bar{z}}{a\bar{x} + b\bar{z}}\right)\hat{z}$$

Examples

- Combine these rules to log-linearize more complex expressions

$$y = f(x) = (ax + b)^c$$

- Gives

$$\hat{y} = c\left(\frac{a\bar{x}}{a\bar{x} + b}\right)\hat{x}$$

Ramsey-Cass-Koopmans growth model

- Recall dynamical system in consumption c_t and capital k_t

$$u'(c_t) = \beta u'(c_{t+1})\big[f'(k_{t+1}) + 1 - \delta\big] \qquad \text{(Euler eq.)}$$

and

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \qquad \text{(resource constraint)}$$

with given initial condition and transversality condition

- Next steps: (i) log-linearization around steady state $\bar{c}, \bar{k}$ then (ii) determine magnitudes of eigenvalues, stability

Log-linearizing the growth model

- Resource constraint in levels

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- Log-linearized version, treated as exact

$$\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} = f'(\bar{k})\bar{k}\hat{k}_t + (1-\delta)\bar{k}\hat{k}_t$$

- Recall that in steady state

$$1 = \beta\left[f'(\bar{k}) + 1 - \delta\right]$$

- Using this to simplify, gives

$$\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} = \frac{1}{\beta}\bar{k}\hat{k}_t$$

Log-linearizing the growth model

- Consumption Euler equation in levels

$$u'(c_t) = \beta u'(c_{t+1})R_{t+1}, \qquad R_{t+1} \equiv f'(k_{t+1}) + 1 - \delta$$

- Log-linearized version, treated as exact

$$u''(\bar{c})\bar{c}\hat{c}_t = \beta u''(\bar{c})\bar{R}\bar{c}\hat{c}_{t+1} + \beta u'(\bar{c})\bar{R}\hat{R}_{t+1}$$

where

$$\bar{R}\hat{R}_{t+1} = f''(\bar{k})\bar{k}\hat{k}_{t+1} \qquad \text{and} \qquad \beta\bar{R} = 1$$

- Notation: the Arrow/Pratt measure of *relative risk aversion*

$$\sigma(c) \equiv -\frac{u''(c)c}{u'(c)} > 0$$

Log-linearizing the growth model

- With this notation

$$\hat{c}_t = \hat{c}_{t+1} - \frac{1}{\sigma(\bar{c})}\,\hat{R}_{t+1}$$

or

$$\hat{c}_{t+1} - \hat{c}_t = \frac{1}{\sigma(\bar{c})}\,\hat{R}_{t+1}, \qquad \hat{R}_{t+1} = \beta f''(\bar{k})\bar{k}\hat{k}_{t+1}$$

- Note consumption growing, $\hat{c}_{t+1} > \hat{c}_t$, when return on capital is relatively high, $\hat{R}_{t+1} > 0$, i.e., when capital stock will be below steady state, $\hat{k}_{t+1} < 0$ (just as in the phase diagram)

- The coefficient $\frac{1}{\sigma(\bar{c})}$ is a measure of the *intertemporal elasticity of substitution* (willingness to substitute consumption over time)

Log-linearized growth model

- In short, pair of equations

$$\hat{c}_{t+1} = \hat{c}_t + \frac{\beta f''(\bar{k})\bar{k}}{\sigma(\bar{c})}\hat{k}_{t+1}$$

and

$$\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} = \frac{1}{\beta}\bar{k}\hat{k}_t$$

- Implies a system of difference equations

$$\begin{pmatrix} \hat{c}_{t+1} \\ \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 - \frac{\beta f''(\bar{k})\bar{c}}{\sigma(\bar{c})} & \frac{f''(\bar{k})\bar{k}}{\sigma(\bar{c})} \\ \\ -\frac{\bar{c}}{\bar{k}} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \\ \hat{k}_t \end{pmatrix}$$

What are the eigenvalues of this system?

Eigenvalues of the growth model

- Recall that trace of coefficient matrix $\boldsymbol{A}$ is sum of eigenvalues

$$\text{tr}(\boldsymbol{A}) = \lambda_1 + \lambda_2$$

and determinant of $\boldsymbol{A}$ is product of eigenvalues

$$\det(\boldsymbol{A}) = \lambda_1 \, \lambda_2$$

- Gives

$$\lambda_1 + \lambda_2 = 1 - \frac{\beta f''(\bar{k})\bar{c}}{\sigma(\bar{c})} + \frac{1}{\beta} > 2$$

and

$$\lambda_1 \, \lambda_2 = \frac{1}{\beta} > 1$$

- Hence both roots positive and at least one is explosive

Eigenvalues of the growth model

- Recall characteristic polynomial can be written

$$p(\lambda) = \lambda^2 - \text{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

- Now consider polynomial $p(\lambda)$ evaluated at $\lambda = 1$

$$p(1) = (1-\lambda_1)(1-\lambda_2) > 0 \quad \Leftrightarrow \quad \text{both roots on same side of } +1$$

- For the growth model we have

$$p(1) = 1 - \left(1 - \frac{\beta f''(\bar{k})\bar{c}}{\sigma(\bar{c})} + \frac{1}{\beta}\right) + \frac{1}{\beta} = \frac{\beta f''(\bar{k})\bar{c}}{\sigma(\bar{c})} < 0$$

- Hence one stable root $0 < \lambda_1 < 1$, one explosive root $\lambda_2 > 1/\beta > 1$

Implications for stability

- As we saw in the phase diagram, system is *saddle path* unstable
- For almost all initial conditions, system diverges from steady state
- Initial capital $\hat{k}_0$ pre-determined, given exogenously
- Initial consumption $\hat{c}_0$ *not* pre-determined, jumps to *stable arm*
- Compute jump by looking at the solution for the linearized system

Implications for stability

- Recall

$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

hence using $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ and iterating forward

$$\begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix} = \boldsymbol{V}\boldsymbol{\Lambda}^t\boldsymbol{V}^{-1} \begin{pmatrix} \hat{c}_0 \\ \hat{k}_0 \end{pmatrix}$$

- Writing this out explicitly

$$\hat{c}_t = v_{11}\frac{v_{22}\hat{c}_0 - v_{12}\hat{k}_0}{v_{11}v_{22} - v_{12}v_{21}}\lambda_1^t - v_{12}\frac{v_{21}\hat{c}_0 - v_{11}\hat{k}_0}{v_{11}v_{22} - v_{12}v_{21}}\lambda_2^t$$

and

$$\hat{k}_t = v_{21}\frac{v_{22}\hat{c}_0 - v_{12}\hat{k}_0}{v_{11}v_{22} - v_{12}v_{21}}\lambda_1^t - v_{22}\frac{v_{21}\hat{c}_0 - v_{11}\hat{k}_0}{v_{11}v_{22} - v_{12}v_{21}}\lambda_2^t$$

Initial jump $\hat{c}_0$

- If λ_2 is the unstable root, then setting

$$\hat{c}_0 = \frac{v_{11}}{v_{21}}\,\hat{k}_0$$

will neutralize the explosive dynamics

- Plugging this jump for initial consumption gives

$$\hat{c}_t = v_{11}\frac{v_{22}\frac{v_{11}}{v_{21}} - v_{12}}{v_{11}v_{22} - v_{12}v_{21}}\lambda_1^t\hat{k}_0 = \frac{v_{11}}{v_{21}}\,\lambda_1^t\hat{k}_0$$

and similarly for capital accumulation

$$\hat{k}_t = v_{21}\frac{v_{22}\frac{v_{11}}{v_{21}} - v_{12}}{v_{11}v_{22} - v_{12}v_{21}}\lambda_1^t\hat{k}_0 = \lambda_1^t\hat{k}_0$$

Simple representation

- Let $\lambda \equiv \lambda_1$ denote the stable root

- Solution for capital accumulation

$$\hat{k}_{t+1} = \lambda \hat{k}_t, \qquad t = 0, 1, \ldots, \qquad \hat{k}_0 \text{ given}$$

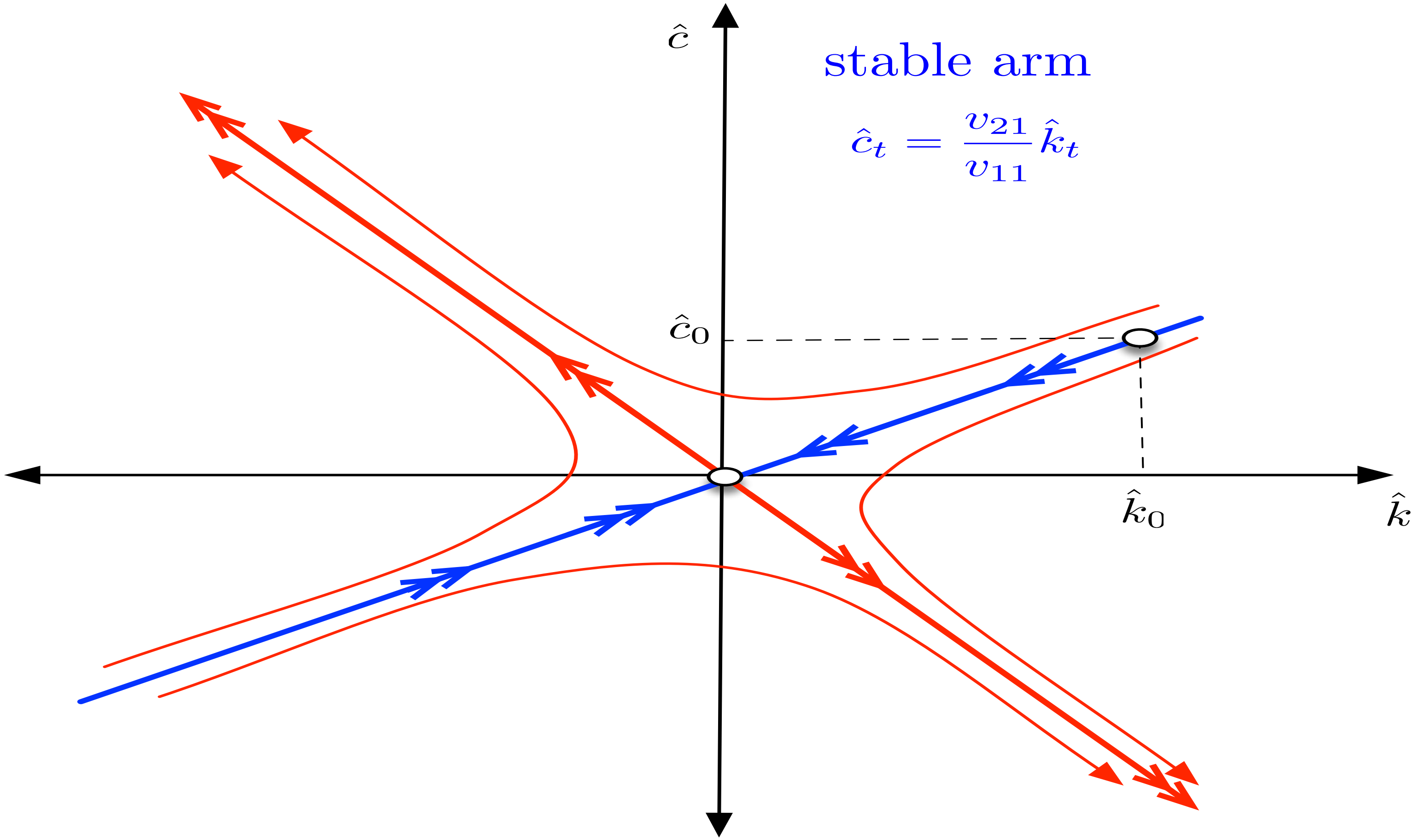
- Solution for consumption, inherits dynamics of capital

$$\hat{c}_t = \frac{v_{11}}{v_{21}} \hat{k}_t, \qquad t = 0, 1, \ldots$$

  This function *is* the stable arm of the saddle path

- Both capital and consumption converge monotonically to steady-state, faster convergence the smaller is $|\lambda|$

Saddle path in log-deviations



Method of undetermined coefficients

- Direct approach that will be useful throughout course

- Guess linear law of motion for capital

$$\hat{k}_{t+1} = \psi_{kk}\,\hat{k}_t$$

and likewise for consumption

$$\hat{c}_t = \psi_{ck}\,\hat{k}_t$$

- Two coefficients that have to be determined

$$\psi_{kk}\,,\,\psi_{ck}$$

Method of undetermined coefficients

- Resource constraint

$$\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} = \frac{1}{\beta}\bar{k}\hat{k}_t$$

- Plug in guesses and collect terms

$$\left[\bar{c}\psi_{ck} + \bar{k}\psi_{kk} - \frac{1}{\beta}\bar{k}\right]\hat{k}_t = 0$$

- This has to hold for *any* value of $\hat{k}_t$. Gives condition

$$\boxed{\bar{c}\psi_{ck} + \bar{k}\psi_{kk} - \frac{1}{\beta}\bar{k} = 0}$$

Method of undetermined coefficients

- Consumption Euler equation

$$\hat{c}_{t+1} = \hat{c}_t + \frac{\beta f''(\bar{k})\bar{k}}{\sigma(\bar{c})}\hat{k}_{t+1}$$

- Plug in guesses and collect terms

$$\left[\psi_{ck}\psi_{kk} - \psi_{ck} - \frac{\beta f''(\bar{k})\bar{k}}{\sigma(\bar{c})}\psi_{kk}\right]\hat{k}_t = 0$$

- This has to hold for *any* value of $\hat{k}_t$. Gives condition

$$\boxed{\psi_{ck}\psi_{kk} - \psi_{ck} - \frac{\beta f''(\bar{k})\bar{k}}{\sigma(\bar{c})}\psi_{kk} = 0}$$

- Two equations in two unknowns ψ_{ck}, ψ_{kk}

Method of undetermined coefficients

- Use first condition to solve for ψ_{ck} in terms of ψ_{kk}

$$\psi_{ck} = \left(\frac{1}{\beta} - \psi_{kk}\right)\frac{\bar{k}}{\bar{c}}$$

- Then plug this into second condition and rearrange to get a *quadratic* in ψ_{kk}

$$\psi_{kk}^2 - \left(1 - \frac{\beta f''(\bar{k})\bar{c}}{\sigma(\bar{c})} + \frac{1}{\beta}\right)\psi_{kk} + \frac{1}{\beta} = 0$$

- Looks familiar. This is *exactly the same* characteristic polynomial we had before. Two solutions, one stable and one unstable

Solution

- In short, the coefficient ψ_{kk} <u>*is*</u> the stable eigenvalue

$$\psi_{kk} = \lambda \in (0, 1)$$

Can then recover the slope of the stable-arm

$$\psi_{ck} = \left(\frac{1}{\beta} - \psi_{kk}\right)\frac{\bar{k}}{\bar{c}} > 0$$

- So far, have simply confirmed the intuition in the phase diagram
- But want to go further than that, want to actually solve the model
- To do this, need specific functional forms and parameter values
- Can then calculate ψ_{kk} , ψ_{ck} and study dynamics

Numerical example

- Suppose standard functional forms

$$f(k) = k^{\alpha}, \qquad u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

with parameter values $\alpha = 0.3$, $\sigma = 1$ and $\rho = \delta = 0.05$ (annual)

- Implies steady state values

$$\frac{\bar{k}}{\bar{y}} = \frac{\alpha}{\rho + \delta} = 3, \qquad \frac{\bar{c}}{\bar{y}} = \frac{\rho + (1-\alpha)\delta}{\rho + \delta} = 0.85$$

and

$$f''(\bar{k})\bar{k} = -(1-\alpha)(\rho + \delta) = -0.07$$

Numerical example

- Use these steady state values to get coefficient matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 - \frac{\beta f''(\bar{k})\bar{c}}{\sigma} & \frac{f''(\bar{k})\bar{k}}{\sigma} \\ \\ -\frac{\bar{c}}{\bar{k}} & \frac{1}{\beta} \end{pmatrix} = \begin{pmatrix} 1.02 & -0.07 \\ \\ -0.28 & 1.05 \end{pmatrix}$$

- Trace

$$\text{tr}(\boldsymbol{A}) = 1.02 + 1.05 = 2.07$$

- Determinant

$$\det(\boldsymbol{A}) = (1.02)(1.05) - (0.07)(0.28) = 1.05$$

Numerical example

- Eigenvalues are roots of characteristic polynomial

$$p(\lambda) = \lambda^2 - \text{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) = \lambda^2 - (2.07)\lambda + 1.05 = 0$$

- Solving this gives

$$\lambda_1 = 0.89, \qquad \lambda_2 = 1.18$$

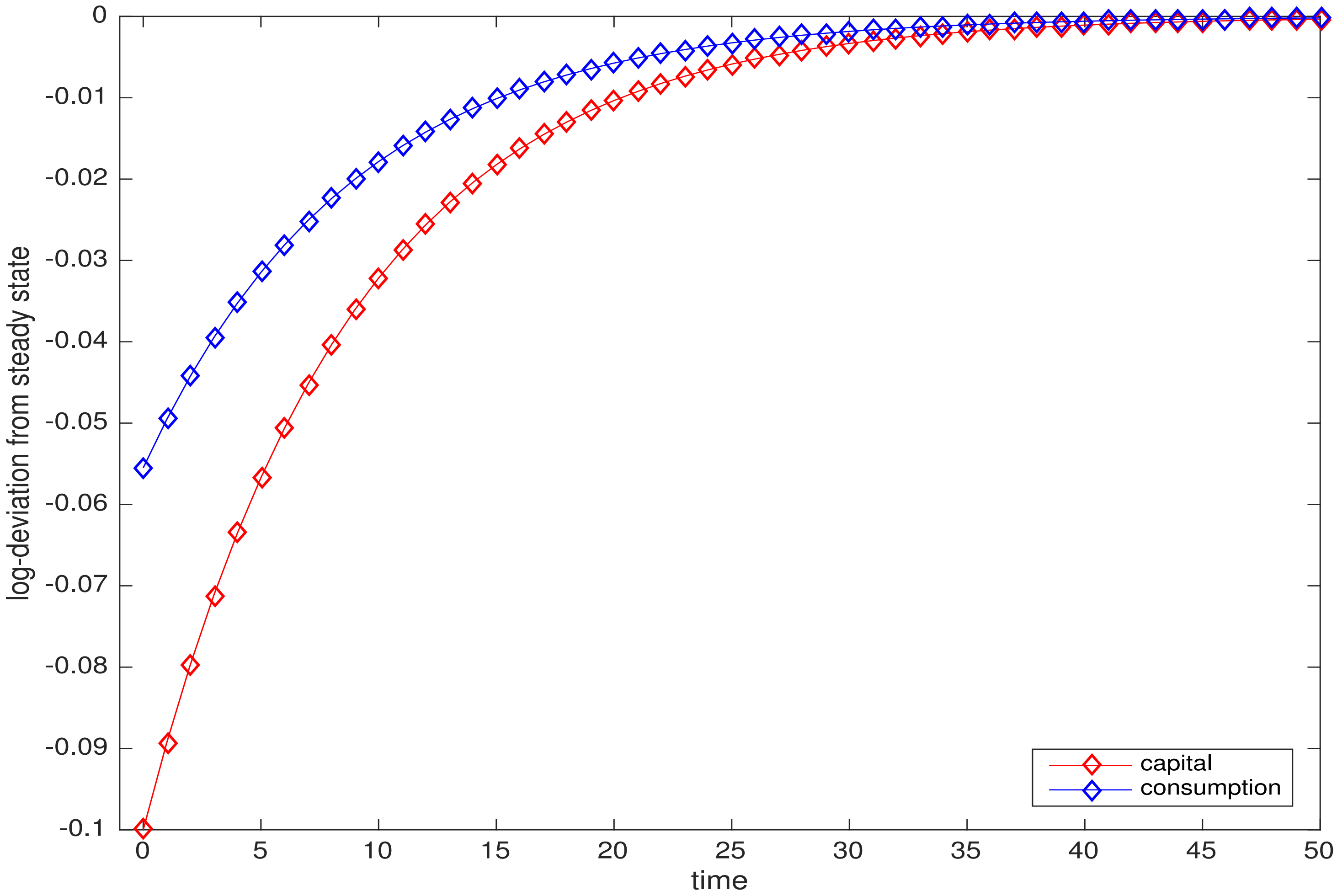
- Choose stable root

$$\psi_{kk} = \lambda_1 = 0.89$$

- Implies slope of the stable arm

$$\psi_{ck} = \left(\frac{1}{\beta} - \psi_{kk}\right)\frac{\bar{k}}{\bar{c}} = \frac{1.05 - 0.89}{0.28} = 0.56$$

Transition to steady state



Initial capital $\hat{k}_0 = -0.1$ (i.e., 10% below steady state). Capital $\hat{k}_{t+1} = \psi_{kk}\hat{k}_t$ and consumption $\hat{c}_t = \psi_{ck}\hat{k}_t$ with $\psi_{kk} = 0.89$ and $\psi_{ck} = 0.56$.

Scraps of Matlab code

From Matlab script "*optimal_growth_example.m*" in LMS

```
%%%%% parameters

alpha = 0.30;      %% capital's share
beta  = 1/1.05;    %% discount factor
delta = 0.05;      %% depreciation rate
sigma = 1;         %% coefficient relative risk aversion
```

```
%%%%% steady state

rho    = 1/beta - 1;                      %% discount rate

r      = rho+delta;                       %% steady-state mpk

kbar   = (alpha/r)^(1/(1-alpha));    %% steady-state k
ybar   = kbar^alpha              ;    %% steady-state y
cbar   = ybar - delta*kbar;           %% steady-state c

d2f    = alpha*(alpha-1)*kbar^(alpha-2); %% f''(kbar)
```

```
%%%%% coefficient matrix

AA = [ 1-beta*d2f*cbar/sigma , d2f*kbar/sigma;
         -cbar/kbar            , 1/beta             ];
```

```
%%%%% eigenvalues are roots of quadratic in lambda

lambdas = roots([1,-trace(AA),det(AA)]);
```

```
%%%%%% choose stable root

lambda  = min(abs(lambdas));

if abs(lambda)>1,
    display('check roots')
end

%%%%%% solution

psi_kk  = lambda;
psi_ck  = (1/beta - psi_kk)*kbar/cbar;
```

Transition path from initial condition

```
%%%%% transitional dynamics

k0       = -0.2; %% initial k, log-dev from steady state

T        = 51;   %% horizon

kt       = zeros(T,1);
ct       = zeros(T,1);

kt(1)    = k0;
ct(1)    = psi_ck*k0;

for t=1:T-1,

    kt(t+1) = psi_kk*kt(t);
    ct(t+1) = psi_ck*kt(t+1);

end
```

Simple plot

```
time = (0:1:T-1); %% time index

figure(1)
plot(time,kt,'rd-',time,ct,'bd-')
ylabel('log-deviation from steady state')
xlabel('time')
axis([min(time) max(time) k0 0])
legend('capital','consumption','location','SouthEast')
```

Next class

- Ramsey-Cass-Koopmans growth model in continuous time
  - brief introduction to optimal control theory
  - decentralization of planning problem