Advanced Macroeconomics

Lecture 3: growth theory and dynamic optimization, part two

Chris Edmond

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This class

- Ramesy-Cass-Koopmans *optimal* growth model in discrete time
  - optimal savings, not an exogenous constant
  - intertemporal utility maximization

  - characterization of solution, two-dimensional phase diagram
  - saddle-path dynamics (sketch)

Ramesy-Cass-Koopmans growth model

- Optimal savings, not an exogenous constant

- Solve the problem of a '*benevolent social planner*'

  – how should society save?

  – in the absence of frictions, the outcome chosen by the social planner can generally be implemented using market arrangements (a version of the second welfare theorem)

  – will see how to do this *decentralization* later

Setup

- Discrete time $t = 0, 1, 2, \ldots$

- Aggregate production function

$$Y_t = F(K_t, L)$$

(for now, keep things simply by setting $A_t = 1$ and $L_t = L$)

- Physical capital depreciates at rate δ

$$K_{t+1} = (1-\delta)K_t + I_t, \qquad 0 < \delta < 1, \qquad K_0 > 0$$

- Goods may be either consumed or invested

$$C_t + I_t = Y_t$$

- Gives the sequence of *resource constraints*, one for each date

$$C_t + K_{t+1} = F(K_t, L) + (1-\delta)K_t, \qquad K_0 > 0$$

Intensive form

- In per worker units

$$y \equiv \frac{Y}{L}, \qquad k \equiv \frac{K}{L}, \ldots \qquad \text{etc}$$

- Aggregate production function

$$y = f(k)$$

- Resource constraints

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \qquad k_0 > 0$$

Intertemporal utility

- Social planner seeks to maximize intertemporal utility

$$U\left(\{c_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right), \qquad 0 < \beta < 1$$

with strictly concave period utility $u'(c) > 0$, $u''(c) < 0$

- Future is discounted by constant factor β

$$1, \quad \beta, \quad \beta^2, \quad \beta^3, \quad \ldots$$

- $U(\cdot)$ is *time-separable*, marginal utility of date-t consumption

$$\frac{\partial U(\cdot)}{\partial c_t} = \beta^t u'(c_t)$$

depends only on c_t, not consumption on other dates

- Infinite horizon keeps model 'stationary', no life-cycle effects

Social planner's problem

- Lagrangian with multiplier $\lambda_t \geq 0$ for each resource constraint

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \big[f(k_t) + (1-\delta)k_t - c_t - k_{t+1}\big]$$

- Some key *first order conditions*

$$c_t : \qquad \beta^t u'(c_t) - \lambda_t = 0$$

$$k_{t+1} : \qquad -\lambda_t + \lambda_{t+1}\big[f'(k_{t+1}) + 1 - \delta\big] = 0$$

$$\lambda_t : \qquad f(k_t) + (1-\delta)k_t - c_t - k_{t+1} = 0$$

These hold at every date

Key intertemporal condition

- Eliminating the Lagrange multipliers

$$u'(c_t) = \beta u'(c_{t+1})\left[f'(k_{t+1}) + 1 - \delta\right]$$

(the '*consumption Euler equation*')

- MRS between t and $t+1$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})}$$

MRT between t and $t+1$

$$f'(k_{t+1}) + 1 - \delta$$

Planner equates MRS and MRT

Dynamical system

- Gives a system of two nonlinear difference equations in c_t, k_t

$$u'(c_t) = \beta u'(c_{t+1})\big[f'(k_{t+1}) + 1 - \delta\big]$$

and

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- Two boundary conditions: (i) initial $k_0 > 0$ given, and (ii) the '*transversality condition*'

$$\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0$$

(analogous to $k_{T+1} = 0$ we would have in finite-horizon model)

Steady state

- Steady state where $\Delta c_t = 0$ and $\Delta k_t = 0$. Let c^*, k^* denote steady state values. These are determined by

  $$1 = \beta\left[f'(k^*) + 1 - \delta\right]$$

  and

  $$c^* + k^* = f(k^*) + (1 - \delta)k^*$$

- Steady state Euler equation pins down k^*, resource constraint then determines c^*, in particular

  $$c^* = f(k^*) - \delta k^*$$

Modified golden rule

- Let $C(k)$ denote consumption sustained by holding k_t fixed at k

$$C(k) \equiv f(k) - \delta k$$

- $C(k)$ is maximized at the '*golden rule*' level, where

$$f'(k) = \delta$$

(remember that in this simplified setup $g = n = 0$)

- Steady state capital stock determined by

$$f'(k) = \rho + \delta, \qquad \rho \equiv \frac{1}{\beta} - 1 > 0$$

where $\rho > 0$ is the pure *rate of time preference*

- Hence steady state capital is less than the golden rule level

Qualitative dynamics

- Consumption dynamics

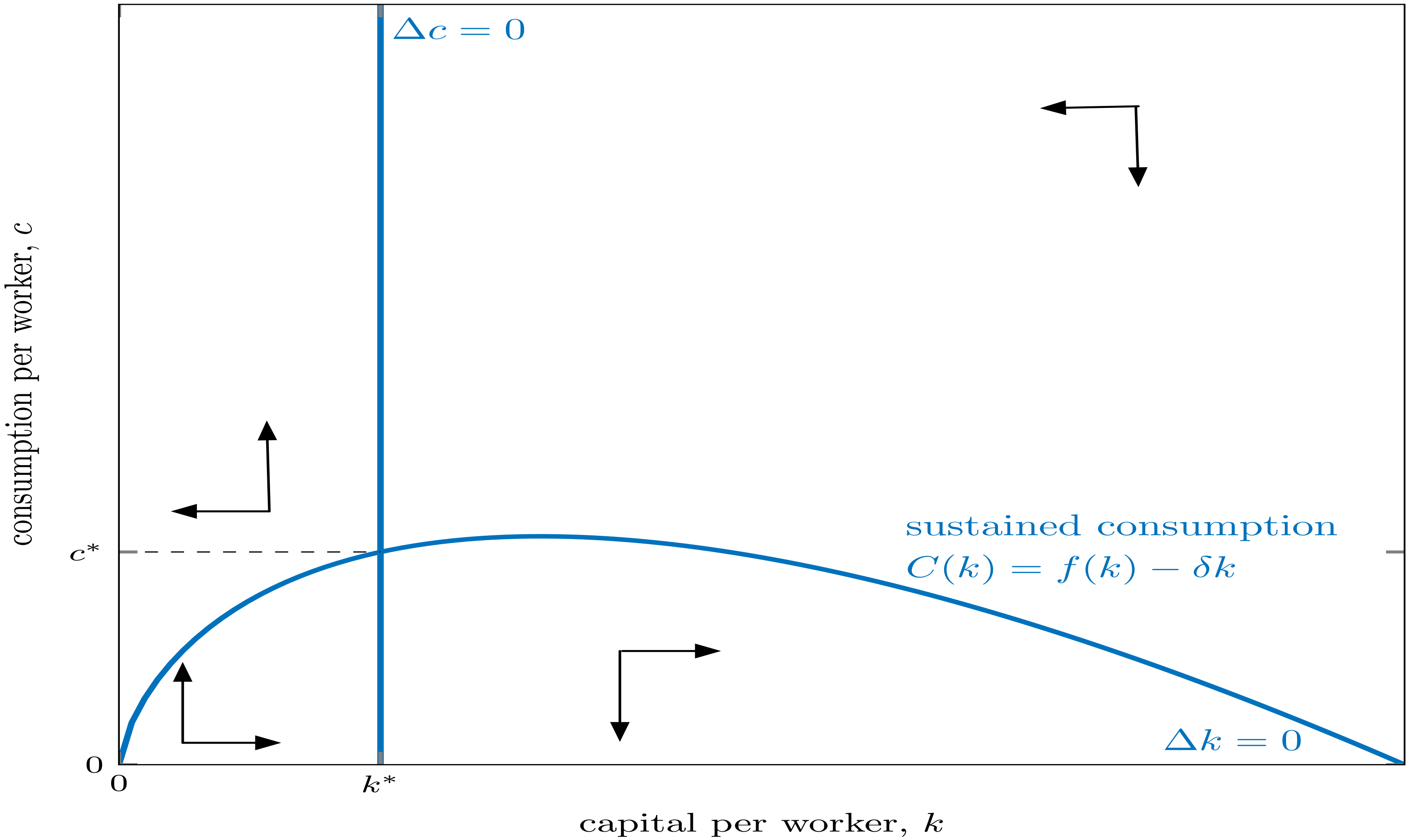
$$c_{t+1} > c_t \qquad \Leftrightarrow \qquad k_{t+1} < k^*$$

- Capital dynamics

$$k_{t+1} > k_t \qquad \Leftrightarrow \qquad c_t < C(k_t)$$

- Divides k_t, c_t space into *four regions*. Flows can be analyzed with a two-dimensional phase diagram

Phase diagram in k_t, c_t space



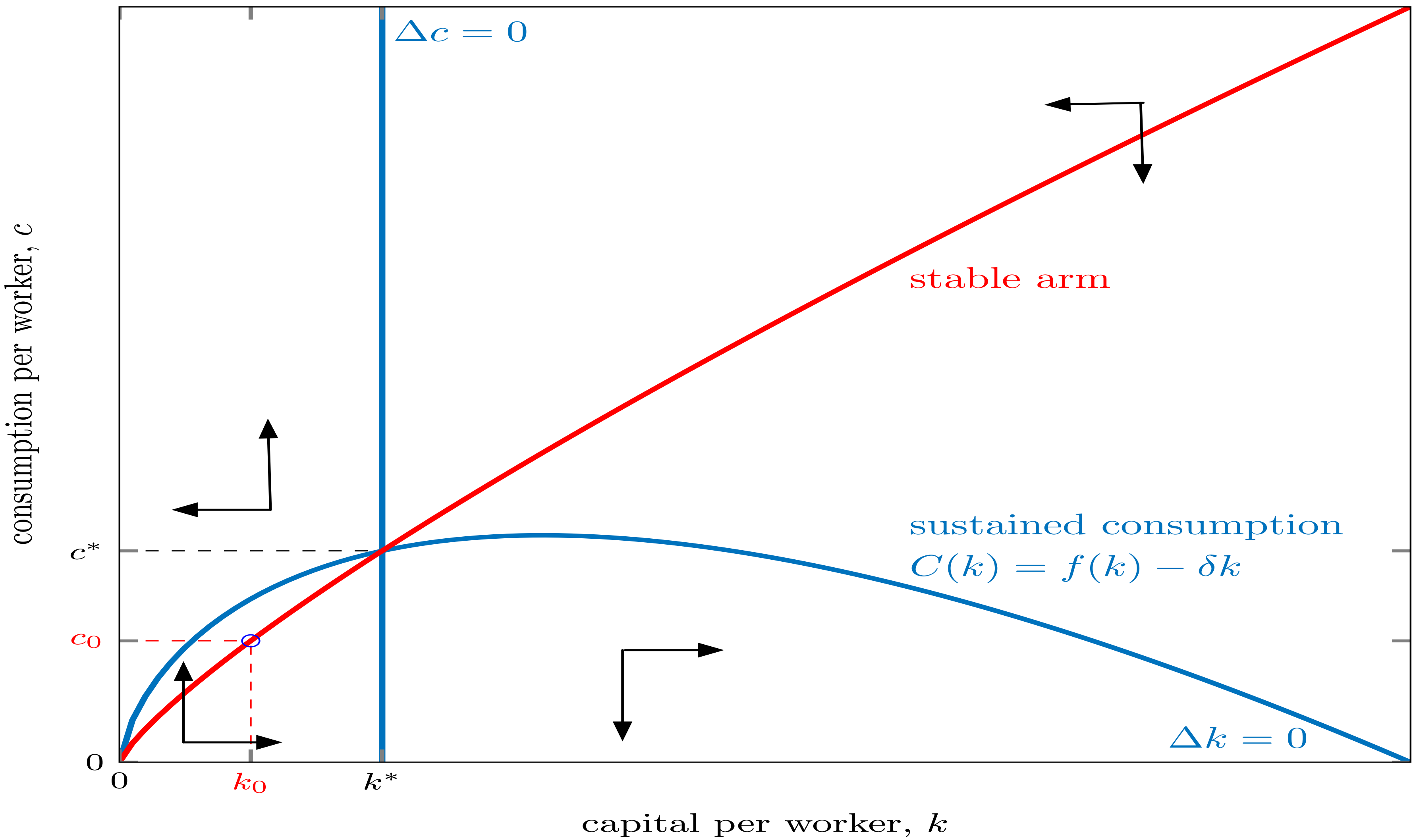
Determining c_0

- Capital k_0 is *pre-determined* (historically given) at date $t = 0$
- Consumption c_0 not pre-determined, can '*jump*' within feasible set

$$0 \leq c_0 \leq C(k_0) + k_0$$

- Consumption c_0 jumps to '*stable arm*' of the dynamical system
- Initial consumption is the one *degree of freedom* that can be used to avoid undesirable trajectories

Stable arm



$\Delta c = 0$
stable arm
sustained consumption
$C(k) = f(k) - \delta k$
$\Delta k = 0$
consumption per worker, c
c^*
c_0
0
0
k_0
k^*
capital per worker, k

Next class

- Stability of *systems* of difference equation
  - eigenvalues, eigenvectors etc
  - 'diagonalizing' systems of difference equations
  - implications for stability of dynamic systems
- Will then start to put these tools to work