Advanced Macroeconomics

Lecture 2: growth theory and dynamic optimization, part one

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This class

- Solow-Swan model in *continuous time*
  - makes for simpler calculations
  - greater transparency in calibration
- Implications and applications
  - balanced growth path
  - long-run effects of changes in savings rate
  - golden rule
  - speed of convergence
  - examples

Towards continuous time

- Period length $\Delta > 0$ in units of calendar time
- Periods $t = 0, \Delta, 2\Delta, 3\Delta, \ldots$
- All *flows* multiplied by period length, so for example

$$K_{t+\Delta} - K_t = I_t\Delta - \delta\Delta K_t$$

and

$$A_{t+\Delta} = e^{g\Delta} A_t$$

$$L_{t+\Delta} = e^{n\Delta} L_t$$

(in anticipation of continuously-compounded growth rates)

Towards continuous time

- Divide by $\Delta > 0$

$$\frac{K_{t+\Delta} - K_t}{\Delta} = I_t - \delta K_t$$

and

$$\frac{A_{t+\Delta} - A_t}{\Delta} = \frac{e^{g\Delta} - 1}{\Delta} A_t$$

$$\frac{L_{t+\Delta} - L_t}{\Delta} = \frac{e^{n\Delta} - 1}{\Delta} L_t$$

- We now want to take limit as period length shrinks $\Delta \to 0$. Using l'Hôpital's rule,

$$\lim_{\Delta \to 0} \frac{e^{x\Delta} - 1}{\Delta} = x$$

(or can use $e^{x\Delta} \approx 1 + x\Delta$)

Continuous time limit

- Gives

$$\dot{K}(t) \equiv \frac{dK(t)}{dt} = I(t) - \delta K(t)$$

and

$$\dot{A}(t) \equiv \frac{dA(t)}{dt} = gA(t), \qquad \dot{L}(t) \equiv \frac{dL(t)}{dt} = nL(t)$$

- Productivity $A(t)$ and the labor force $L(t)$ grow exponentially

$$\frac{\dot{A}(t)}{A(t)} = g \quad \Rightarrow \quad A(t) = e^{gt}A(0)$$

$$\frac{\dot{L}(t)}{L(t)} = n \quad \Rightarrow \quad L(t) = e^{nt}L(0)$$

Solow-Swan in continuous time

- Time $t \geq 0$
- Capital accumulation

$$\dot{K}(t) = I(t) - \delta K(t)$$

- Exogenous productivity and labor force

$$\dot{A}(t) = gA(t), \qquad \dot{L}(t) = nL(t)$$

- Constant savings rate

$$I(t) = S(t) = sY(t) = sF(K(t), A(t)L(t))$$

- Aggregate production function $Y = F(K, AL)$ satisfying the usual assumptions

Solow-Swan in continuous time

- Hence

$$\dot{K}(t) = sF(K(t), A(t)L(t)) - \delta K(t)$$

- Define intensive variables as usual $k \equiv K/AL$, $y \equiv Y/AL$, $y = f(k)$ etc and note

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)}$$

- Hence in intensive form

$$\dot{k}(t) = sf(k(t)) - (\delta + g + n)k(t) \equiv \psi(k(t))$$

An autonomous nonlinear *differential equation* in $k(t)$ with transition function $\psi(k)$

Solow-Swan in continuous time

- Steady state k^* where $\dot{k}(t) = 0$, i.e., solves usual condition

$$sf(k^*) = (\delta + g + n)k^*$$

- At k^*, $sf'(k^*)$ is less than $\delta + g + n$, i.e., $\psi'(k^*) < 0$ [why?]

- Qualitative dynamics

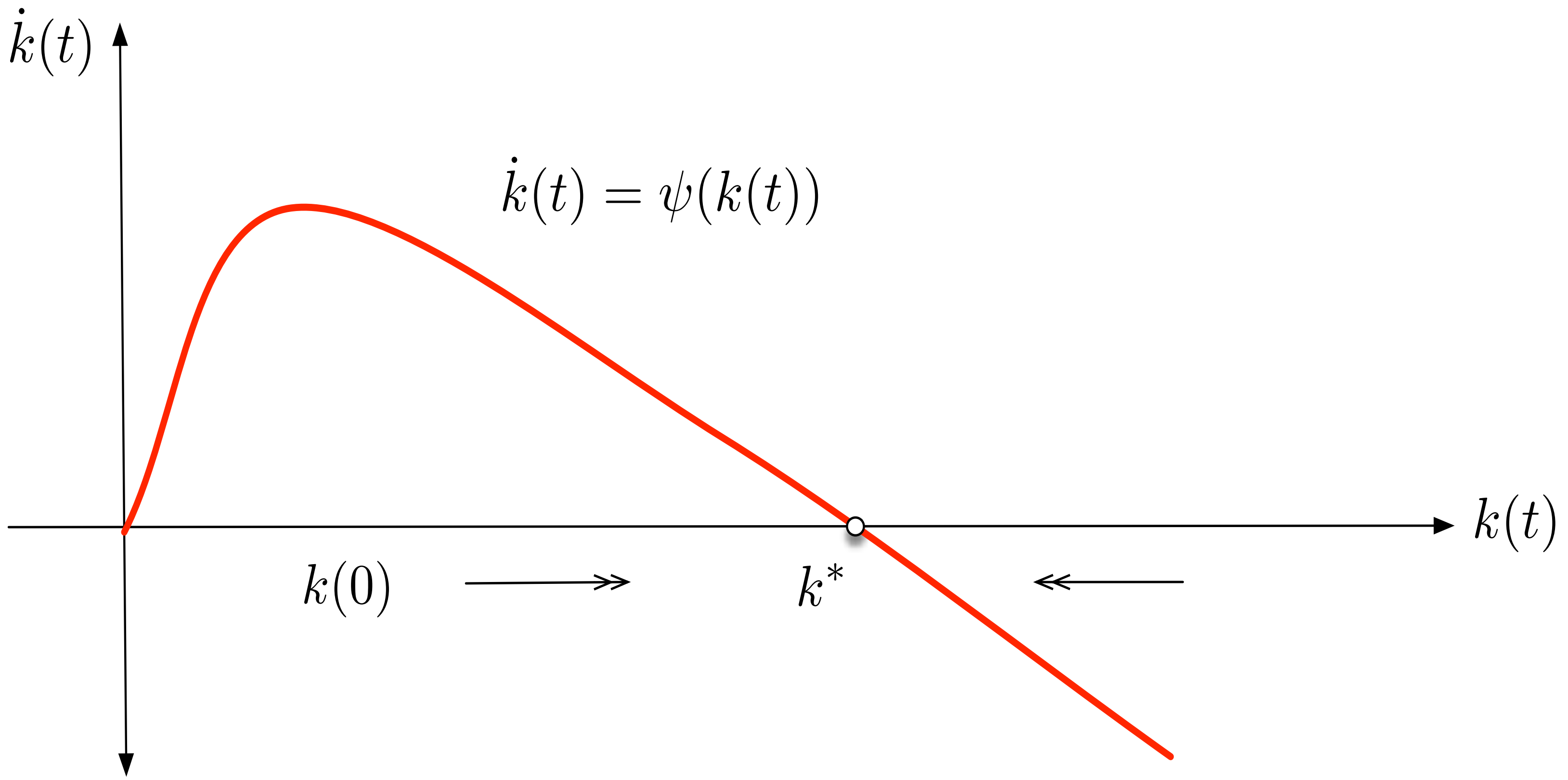
$$\begin{aligned}\dot{k}(t) > 0 \quad &\Leftrightarrow \quad sf(k(t)) > (\delta + g + n)k(t) \\ &\Leftrightarrow \quad k(t) < k^*\end{aligned}$$

and

$$\begin{aligned}\dot{k}(t) < 0 \quad &\Leftrightarrow \quad sf(k(t)) < (\delta + g + n)k(t) \\ &\Leftrightarrow \quad k(t) > k^*\end{aligned}$$

- Converges $k(t) \rightarrow k^*$, steady state k^* is *stable* (for all $k(0) > 0$).

Phase diagram



Linear differential equation

- To understand these stability properties more systematically, let's begin with simple scalar linear differential equations, such as

$$\dot{x}(t) = \lambda x(t) + b, \qquad x(0) \text{ given}$$

- Steady state if $\lambda \neq 0$

$$x^* = -\frac{b}{\lambda}$$

- Can then write in *deviations* from steady state

$$\dot{x}(t) = \lambda(x(t) - x^*)$$

Notice that $\dot{x}(t) \equiv \frac{d}{dt}x(t) = \frac{d}{dt}(x(t) - x^*)$ so $\dot{x}(t)$ is also the time derivative of the deviation from steady state

Linear differential equation

- Stability properties determined by *magnitude* of coefficient λ

- If $\lambda \neq 0$

$$x(t) = x^* + e^{\lambda t}(x(0) - x^*), \qquad t \geq 0$$

  If $\lambda < 0$ then $x(t)$ *converges* (monotonically) to x^* as $t \to \infty$. If $\lambda > 0$ then $x(t)$ *diverges* to $\pm\infty$ depending on sign of $x(0) - x^*$

- If $\lambda = 0$, no steady state and simply

$$x(t) = tb + x(0)$$

- In short steady state stable if $\lambda < 0$ and unstable otherwise. In linear system, *local* stability implies *global* stability

Nonlinear differential equation

- Consider scalar nonlinear differential equation

$$\dot{x}(t) = \psi(x(t)), \qquad x(0) \text{ given}$$

Steady states determined by

$$0 = \psi(x^*)$$

May be many, or none

- Stability *local* to a steady state depends on sign of $\psi'(x^*)$

- Approximate solution, local to x^*

$$x(t) \approx x^* + e^{\psi'(x^*)t}(x(0) - x^*), \qquad t \geq 0$$

So that $x(t) \to x^*$ if coefficient $\psi'(x^*) < 0$

Approximate Solow-Swan dynamics

- Exact nonlinear differential equation

$$\dot{k}(t) = sf(k(t)) - (\delta + g + n)k(t) \equiv \psi(k(t))$$

- Approximate solution, local to k^*

$$k(t) \approx k^* + e^{\psi'(k^*)t}(k(0) - k^*), \qquad t \geq 0$$

where

$$\psi'(k^*) = sf'(k^*) - (\delta + g + n) < 0$$

- Weighted average of initial $k(0)$ and steady state k^* with weight on $k(0)$ decreasing exponentially

Balanced growth path

- Asymptotically $k(t) \to k^*$ and $y(t) \to y^* = f(k^*)$
- Hence capital $K(t)$, output $Y(t)$ and consumption $C(t)$ all asymptotically grow at $g + n$
- Hence capital per worker $K(t)/L(t)$, output per worker $Y(t)/L(t)$ and consumption per worker $C(t)/L(t)$ all asymptotically grow at g
- Long run growth independent of savings rate s and independent of initial conditions $K(0), A(0), L(0)$.

Changes in savings rate s

- Has *level* effect on k^* and hence on y^* and c^*. How do we formally determine this level effect? Use *comparative statics*

- Recall k^* solves

$$sf(k^*) = (\delta + g + n)k^*$$

- *Implicitly determines* k^* as a function of s. Write $k^* = k(s)$ so

$$sf(k(s)) = (\delta + g + n)k(s)$$

- Differentiate both sides and rearrange

$$k'(s) = -\frac{f(k^*)}{sf'(k^*) - (\delta + g + n)} > 0$$

which is positive since $sf'(k^*) < (\delta + g + n)$

- In short, a permanent increase in the savings rate s permanently increases k^* (but does not change long run growth)

Golden rule

- Consumption $c^* = c(s)$ via

$$c(s) = (1-s)f(k(s)) = f(k(s)) - (\delta + g + n)k(s)$$

An increase in s may increase or decrease $c(s)$

- What level of s maximizes steady state $c^* = c(s)$? First order condition for this problem

$$c'(s) = 0 \qquad \Leftrightarrow \qquad \Big[f'(k) - (\delta + g + n)\Big]k'(s) = 0$$

Since $k'(s) > 0$ this requires

$$f'(k) = \delta + g + n$$

- Choose s to make $k(s) = k^*$ such that $f'(k^*) = \delta + g + n$. Equivalently, so that

$$s = \frac{f'(k^*)k^*}{f(k^*)}$$

Speed of convergence

- Recall that speed of convergence depends on magnitude of

$$\psi'(k^*) = sf'(k^*) - (\delta + g + n) < 0$$

- Can write this

$$\psi'(k^*) = \frac{sf'(k^*)k^*}{k^*} - (\delta + g + n)$$

$$= \frac{f'(k^*)k^*}{f(k^*)}(\delta + g + n) - (\delta + g + n)$$

$$= -\left(1 - \frac{f'(k^*)k^*}{f(k^*)}\right)(\delta + g + n) < 0$$

- Speed of convergence determined by (i) the *degree of concavity* in the production function and (ii) the effective depreciation rate

Cobb-Douglas example

- Suppose aggregate production function

$$Y = F(K, AL) = K^{\alpha}(AL)^{1-\alpha}, \qquad 0 < \alpha < 1$$

so that in intensive form

$$y = f(k) = k^{\alpha}$$

- In this special case, the elasticity of output with respect to capital is constant

$$\frac{f'(k)k}{f(k)} = \alpha$$

Consequently, the golden rule savings rate is just $s = \alpha$

Cobb-Douglas example

- Steady state capital

$$sk^{\alpha} = (\delta + g + n)k \qquad \Rightarrow \qquad k^* = \left(\frac{s}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}$$

- Steady state capital/output ratio

$$\frac{k^*}{y^*} = \frac{s}{\delta + g + n}$$

Cobb-Douglas example

- Amusingly, for the Cobb-Douglas case the nonlinear differential equation can be solved exactly. This relies on a simple trick

- It turns our that in the Cobb-Douglas case the Solow-Swan model implies a *linear* differential equation in the *capital/output ratio*

- Let $x(t)$ denote the capital/output ratio, which in this case is

$$x(t) \equiv \frac{K(t)}{Y(t)} = \frac{k(t)}{y(t)} = k(t)^{1-\alpha}$$

with given initial condition

$$x(0) = k(0)^{1-\alpha} > 0$$

Cobb-Douglas example

- So in this special case we have the linear differential equation

$$\dot{x}(t) = (1-\alpha)\Big[s - (\delta + g + n)x(t)\Big]$$

- This has the exact solution

$$x(t) = e^{\lambda t}x(0) + (1 - e^{\lambda t})x^*, \qquad t \geq 0$$

where the steady state is

$$x^* = \frac{k^*}{y^*} = \frac{s}{\delta + g + n}$$

and where the speed of adjustment is

$$\lambda = -(1-\alpha)(\delta + g + n) < 0$$

(note, this is $\lambda = \psi'(k^*)$, the derivative of the transition function as per slide 17 above)

Cobb-Douglas example

- Can then write the exact solution for $k(t) = x(t)^{\frac{1}{1-\alpha}}$ as

$$k(t) = \left(e^{\lambda t} k(0)^{1-\alpha} + (1 - e^{\lambda t}) k^{*1-\alpha}\right)^{\frac{1}{1-\alpha}}, \qquad t \geq 0$$

- Rapid convergence to k^* when the speed of adjustment coefficient $\lambda = -(1-\alpha)(\delta + g + n)$ has large magnitude

Next class

- Ramesy-Cass-Koopmans *optimal* growth model in discrete time
  - optimal savings, not an exogenous constant
  - intertemporal utility maximization