Advanced Macroeconomics

Lecture 12: real business cycles, part four

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This class

- Permanent shocks in the RBC model
  - random walks, stochastic trends etc
Stationary AR(1) process

- Recall stationary AR(1) process
  \[ x_{t+1} = (1 - \phi)\bar{x} + \phi x_t + \varepsilon_{t+1}, \quad 0 < |\phi| < 1 \]
  with IID normal innovations \( \varepsilon_{t+1} \sim N(0, \sigma_{\varepsilon}^2) \)

- Long run distribution
  \[ x \sim N\left(\bar{x}, \frac{\sigma_{\varepsilon}^2}{1 - \phi^2}\right) \]
  independent of initial condition \( x_0 \)

- What if \( \phi = 1 \)?
Pure random walk

- With $\phi = 1$ the AR(1) becomes a *random walk*

  $$x_{t+1} = x_t + \varepsilon_{t+1}$$

- Iterating forward from initial condition

  $$x_t = x_0 + \sum_{i=1}^{t} \varepsilon_i$$

- Every single shock realization changes the level of $x_t$ one-for-one. For the random walk, shocks are ‘*permanent*’

- By contrast for the stationary AR(1) shocks are ‘*transitory*’
Pure random walk

- Distribution at date $t$ is
  \[ x_t \sim N \left( x_0, \sigma^2 \varepsilon t \right) \]

- Variance linear in $t$ and dependence on $x_0$ does not fade with time

- Does not converge to a limiting distribution as $t \to \infty$
\[ x_{t+1} = x_t + \varepsilon_{t+1} \]

\( x_0 = 0 \) and \( \varepsilon \sim N(0, \sigma^{2}_{\varepsilon}) \) with \( \sigma_{\varepsilon} = 0.015 \).
Random walk with drift

- Suppose

\[ x_{t+1} = \mu + x_t + \varepsilon_{t+1} \]

- Parameter \( \mu \) is the expected change or \textit{drift}

\[ \mathbb{E}_t[\Delta x_{t+1}] = \mu \]

- Iterating forward from initial condition

\[ x_t = x_0 + \mu t + \sum_{i=1}^{t} \varepsilon_i \]

so that

\[ x_t \sim N \left( x_0 + \mu t, \sigma_\varepsilon^2 t \right) \]

- Mean and variance both linear in \( t \)
\[ x_{t+1} = \mu + x_t + \varepsilon_{t+1} \]

\[ x_0 = 0, \mu = 0.02 \text{ and } \varepsilon \sim N(0, \sigma_\varepsilon^2) \text{ with } \sigma_\varepsilon = 0.015. \]
The random walk has a ‘unit root’ $\phi = 1$

Nonstationary in levels $x_t$ but stationary in first differences $\Delta x_t$

A stochastic process is said to be *integrated of order* $d$ or $I(d)$ if it takes $d$ differences to make the process stationary

So here $x_t$ is $I(1)$ and $\Delta x_t$ is $I(0)$
Another example

• Now consider the following AR(2) process

\[ x_{t+1} = (1 - \phi)\mu + (1 + \phi)x_t - \phi x_{t-1} + \varepsilon_{t+1}, \quad 0 < \phi < 1 \]

• Is this process stationary? What are the roots of this process?

• Equivalent to system of the form

\[
\begin{pmatrix}
x_{t+1} \\
x_t
\end{pmatrix} = \ldots + \begin{pmatrix}
1 + \phi \\ 1
\end{pmatrix} \begin{pmatrix}
x_t \\
x_{t-1}
\end{pmatrix} + \ldots
\]

• Process is stationary if ‘system’ has stable dynamics, i.e., eigenvalues of coefficient matrix are less than one in magnitude.
Eigenvalues

- Eigenvalues of $A$ given by roots of the characteristic polynomial

- For the 2-by-2 case

$$p(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$$

- For this specific example

$$\text{tr}(A) = 1 + \phi > 1, \quad \det(A) = \phi \in (0, 1)$$

- Using the quadratic formula, for this example roots evaluate to

$$\lambda_1 = 1, \quad \lambda_2 = \phi$$

- Hence this is also a ‘unit root process’ and is nonstationary
AR(1) in differences

- In fact this process is simply a stationary AR(1) in differences

\[ \Delta x_{t+1} = (1 - \phi)\mu + \phi \Delta x_t + \epsilon_{t+1}, \quad 0 < \phi < 1 \]

- Since \( \Delta x_t \) is \( I(0) \) the process in levels is \( I(1) \)

- Unlike the random walk, this process implies time-variation in expected growth

\[ \mathbb{E}_t[\Delta x_{t+1}] = (1 - \phi)\mu + \phi \Delta x_t \]

with reversion to long-run average growth \( \mu \) governed by \( \phi \)
\[ \Delta x_{t+1} = (1 - \phi) \mu + \phi \Delta x_t + \varepsilon_{t+1} \]

\[ x_0 = 0, \ \mu = 0.02, \ \phi = 0.5 \text{ and } \varepsilon \sim N(0, \sigma_\varepsilon^2) \text{ with } \sigma_\varepsilon = 0.015. \]
Cointegration

• A scalar $x_t$ is $I(1)$ if $\Delta x_t$ is stationary, $I(0)$

• A vector of $I(1)$ variables $\mathbf{x}_t$ is cointegrated if there is a linear combination $\mathbf{a}'\mathbf{x}_t$ that is $I(0)$

• For example, if $x_t$ and $y_t$ are both $I(1)$ and $ax_t + by_t$ is $I(0)$ then $x_t$ and $y_t$ are cointegrated

• Implies that there is a stable long-run relationship between $x_t$ and $y_t$ even if both nonstationary

• Deviations from $ax_t + by_t$ are mean-reverting, leads to ‘error correction’ representation
Stochastic growth

- What happens if we use nonstationary processes like this to drive the stochastic growth model?

- Suppose standard aggregate production function, in levels

\[ Y_t = F(K_t, A_t L) \]

- Labor-augmenting productivity \( A_t \) is stationary in growth rates

\[ g_t \equiv \frac{A_t}{A_{t-1}} \]

- Constant employment \( L > 0 \) for simplicity
Stochastic growth

- Social planner maximizes expected intertemporal utility
  
  $$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{(C_t/L)^{1-\sigma} - 1}{1 - \sigma} \right\}, \quad 0 < \beta < 1, \quad \sigma > 0$$

  subject to sequence of resource constraints, for each date and state

  $$C_t + K_{t+1} = F(K_t, A_t L) + (1 - \delta)K_t, \quad 0 < \delta < 1$$

- Initial $K_0 > 0$ and stochastic process for productivity $\{A_t\}$ given

- Isoelastic utility needed for balanced growth
Intensive form

- In efficiency units
  \[ y_t \equiv \frac{Y_t}{A_t L}, \quad c_t \equiv \frac{C_t}{A_t L}, \quad k_t \equiv \frac{K_t}{A_{t-1} L} \]

- Note capital \( K_t \) divided by lagged productivity \( A_{t-1} \)

- This implies that detrended \( k_t \) remains ‘predetermined’
Detrending

- Resource constraint

\[ C_t + K_{t+1} = F(K_t, A_tL) + (1 - \delta)K_t \]

- Dividing through by \( A_t \) and using \( g_t = A_t/A_{t-1} \) gives

\[ c_t + k_{t+1} = f\left(\frac{k_t}{g_t}\right) + (1 - \delta)\left(\frac{k_t}{g_t}\right) \]

where \( f(\cdot) \) denotes the intensive form of the production function

- Period utility

\[ \frac{(C_t/L)^{1-\sigma} - 1}{1 - \sigma} = \frac{(c_tA_t)^{1-\sigma} - 1}{1 - \sigma} \]

If log utility, \( \sigma \to 1 \), period utility is separable in \( c_t \) and \( A_t \)
Social planner’s problem

• Lagrangian with stochastic multiplier $\lambda_t \geq 0$ for each constraint

$$\mathcal{L} = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{(c_t A_t)^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \left[ f(k_t/g_t) + (1-\delta)(k_t/g_t) - c_t - k_{t+1} \right] \right\}$$

• Some key first order conditions

$$c_t : \quad \beta^t c_t A_t^{1-\sigma} - \lambda_t = 0$$

$$k_{t+1} : \quad -\lambda_t + \mathbb{E}_t \left\{ \lambda_{t+1} \left[ f'(k_{t+1}/g_{t+1})/g_{t+1} + (1-\delta)/g_{t+1} \right] \right\} = 0$$

$$\lambda_t : \quad f(k_t/g_t) + (1-\delta)(k_t/g_t) - c_t - k_{t+1} = 0$$

• Although $k_{t+1}$ has a $t + 1$ subscript, it is chosen conditional on date $t$ information
Dynamical system

- Gives a system of stationary stochastic difference equations

\[ c_t^{-\sigma} = \beta \mathbb{E}_t \left\{ c_{t+1}^{-\sigma} g_{t+1}^{1-\sigma} \left[ f'(k_{t+1}/g_{t+1})/g_{t+1} + (1 - \delta)/g_{t+1} \right] \right\} \]

and

\[ c_t + k_{t+1} = f(k_t/g_t) + (1 - \delta)(k_t/g_t) \]

given initial \( k_0 > 0 \) and transversality condition

- Maps stationary exogenous \( \{g_t\} \) into stationary endogenous \( \{c_t, k_t\} \)

- The term \( \beta g_{t+1}^{1-\sigma} \) in consumption Euler equation is a growth-adjusted discount factor
“Non-stochastic steady state”

- Shut down shocks, set $g_t = \bar{g}$

- Find steady state of associated deterministic model

- Steady state capital $\bar{k}$ solves

$$1 = \beta \bar{g}^{-\sigma} \left[ f'(\bar{k}/\bar{g}) + 1 - \delta \right]$$

- Steady state consumption $\bar{c}$ pinned down by resource constraint

$$\bar{c} = f(\bar{k}/\bar{g}) + (1 - \delta)(\bar{k}/\bar{g}) - \bar{k}$$
Log-linear solution

- Log-linearize the system around these steady state values
  \[
  \hat{k}_t \equiv \log(k_t/\bar{k}), \quad \hat{c}_t \equiv \log(c_t/\bar{c}), \quad \hat{g}_t \equiv \log(g_t/\bar{g})
  \]

- Stationary solution for detrended endogenous variables
  \[
  \hat{k}_{t+1} = \psi_{kk} \hat{k}_t + \psi_{kg} \hat{g}_t
  \]
  and
  \[
  \hat{c}_t = \psi_{ck} \hat{k}_t + \psi_{cg} \hat{g}_t
  \]
  given exogenous stationary process for \( \hat{g}_t \)
Nonstationary variables

- Log-level of productivity

\[ \log A_t = \log A_0 + \sum_{i=1}^{t} \log g_i \]

- Log-levels consumption per worker, capital per worker etc

\[ \log (C_t / L) = \log c_t + \log A_t \]
\[ \log (K_t / L) = \log k_t + \log A_t - \log g_t \]
\[ \log (Y_t / L) = \log y_t + \log A_t \]

- Each is the sum of a stationary component (from the solution of the detrended model) and a common nonstationary component

- Share a common stochastic trend, namely \( \log A_t \)
Cointegration in the growth model

- Log levels of consumption per worker, capital per worker, output per worker etc are all $I(1)$ because of productivity

- Log consumption/output ratio, capital/output ratio etc are $I(0)$. Consumption and output cointegrated, as are capital and output

\[
\log(C_t/Y_t) = \log c_t - \log y_t
\]

\[
\log(K_t/Y_t) = \log k_t - \log y_t - \log g_t
\]

(everything on right hand side is stationary)

- Stable long-run relationships between $C_t, Y_t$ and between $K_t, Y_t$. Deviations from these long-run relationships are mean-reverting
Parameterization

- CRRA utility function [already imposed for balanced growth]

\[ u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma > 0 \]

- Cobb-Douglas production function

\[ f(k) = k^\alpha, \quad 0 < \alpha < 1 \]

- AR(1) process for log productivity growth

\[ \log g_{t+1} = (1 - \phi) \log \bar{g} + \phi \log g_t + \varepsilon_{t+1}, \quad 0 < \phi < 1 \]

with long-run average growth \( \log \bar{g} \) and IID normal innovations

\[ \varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2) \]

- For examples, let \( \sigma = 1 \) and \( \alpha = 0.3, \beta = 0.95, \) and \( \delta = 0.05 \)
Example: pure random walk

- Let $\phi = 0$ with $\bar{g} = 1.00$ and $\sigma_\varepsilon = 0.015$

- No growth but shocks have permanent effect on levels

- Dynare gives coefficients

$$
\begin{pmatrix}
\psi_{kk} & \psi_{kg} \\
\psi_{ck} & \psi_{cg}
\end{pmatrix} =
\begin{pmatrix}
0.89 & -0.89 \\
0.55 & -0.55
\end{pmatrix}
$$
Detrended variables are mean-reverting

Impulse response functions of detrended variables $\hat{c}_t$ and $\hat{k}_t$ to 1 standard deviation shock when log productivity is a pure random walk.
But shocks have permanent effect on levels

Impulse response functions of levels $\log C_t$ and $\log K_t$ to 1 standard deviation shock when $\log A_t$ is a pure random walk.
Example: random walk with drift

- Let $\phi = 0$ with $\bar{g} = 1.02$ and $\sigma_\varepsilon = 0.015$

- Expected growth 2% no matter what current $g_t$ is

- Dynare now gives coefficients

$$
\begin{pmatrix}
\psi_{kk} & \psi_{kg} \\
\psi_{ck} & \psi_{cg}
\end{pmatrix} =
\begin{pmatrix}
0.87 & -0.87 \\
0.54 & -0.54
\end{pmatrix}
$$
Simulated time series for detrended variables $\hat{c}_t$ and $\hat{k}_t$ when log productivity is a random walk with drift.
Simulated time series for levels $\log C_t$ and $\log K_t$ when $\log A_t$ is a random walk with drift. Levels nonstationary but ratios $\log(C_t/Y_t)$ and $\log(K_t/Y_t)$ are stationary.
Example: AR(1) in growth rates

- Let $\phi = 0.5$ with $\bar{g} = 1.02$ and $\sigma_\varepsilon = 0.015$

- Expected growth is time-varying, fluctuates around 2%

- Dynare now gives coefficients

$$
\begin{pmatrix}
\psi_{kk} & \psi_{kg} \\
\psi_{ck} & \psi_{cg}
\end{pmatrix} =
\begin{pmatrix}
0.87 & -0.96 \\
0.54 & -0.28
\end{pmatrix}
$$
Impulse response functions of detrended variables $\hat{c}_t$ and $\hat{k}_t$ to 1 standard deviation shock when log productivity growth is an AR(1).
Simulated time series for levels $\log C_t$ and $\log K_t$ when log productivity growth is an AR(1). Low-frequency fluctuations in stationary ratios more pronounced.
Next class

• Beginning of lectures on monetary economics