Advanced Macroeconomics

Lecture 12: real business cycles, part four

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This class

- Permanent shocks in the RBC model
  - random walks, stochastic trends etc

Stationary AR(1) process

- Recall stationary AR(1) process

$$x_{t+1} = (1-\phi)\bar{x} + \phi x_t + \varepsilon_{t+1}, \qquad 0 < |\phi| < 1$$

with IID normal innovations $\varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2)$

- Long run distribution

$$x \sim N\left(\bar{x}\,,\, \frac{\sigma_\varepsilon^2}{1-\phi^2}\right)$$

independent of initial condition x_0

- What if $\phi = 1$?

Pure random walk

- With $\phi = 1$ the AR(1) becomes a *random walk*

$$x_{t+1} = x_t + \varepsilon_{t+1}$$

- Iterating forward from initial condition

$$x_t = x_0 + \sum_{i=1}^{t} \varepsilon_i$$

- Every single shock realization changes the level of x_t one-for-one. For the random walk, shocks are '*permanent*'

- By contrast for the stationary AR(1) shocks are '*transitory*'

Pure random walk

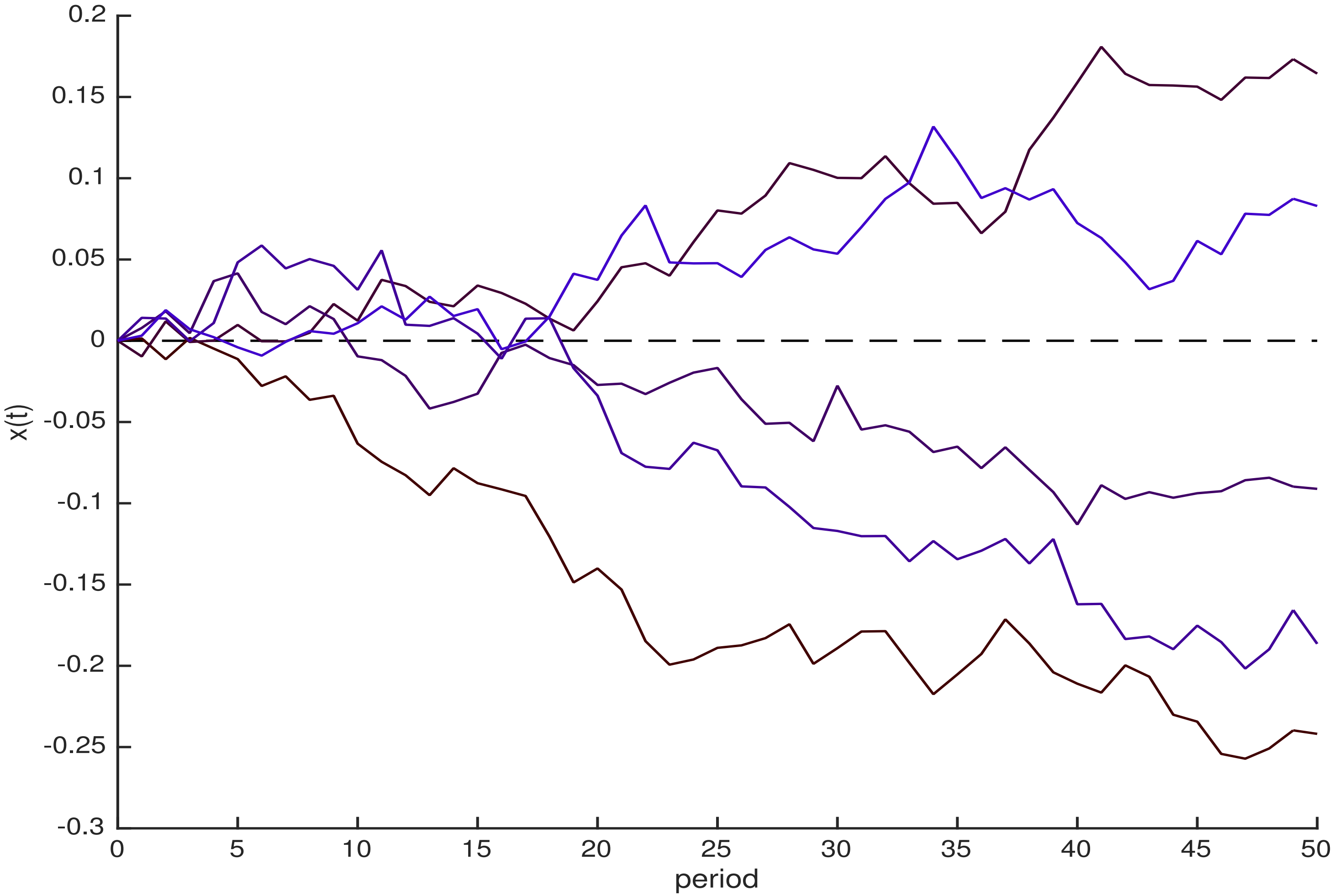
- Distribution at date t is

$$x_t \sim N\left(x_0\,,\,\sigma_\varepsilon^2 t\right)$$

- Variance linear in t and dependence on x_0 does not fade with time

- Does not converge to a limiting distribution as $t \to \infty$

$$x_{t+1} = x_t + \varepsilon_{t+1}$$



$x_0 = 0$ and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$ with $\sigma_\varepsilon = 0.015$.

Random walk with drift

- Suppose

$$x_{t+1} = \mu + x_t + \varepsilon_{t+1}$$

- Parameter μ is the expected change or *drift*

$$\mathbb{E}_t[\Delta x_{t+1}] = \mu$$

- Iterating forward from initial condition

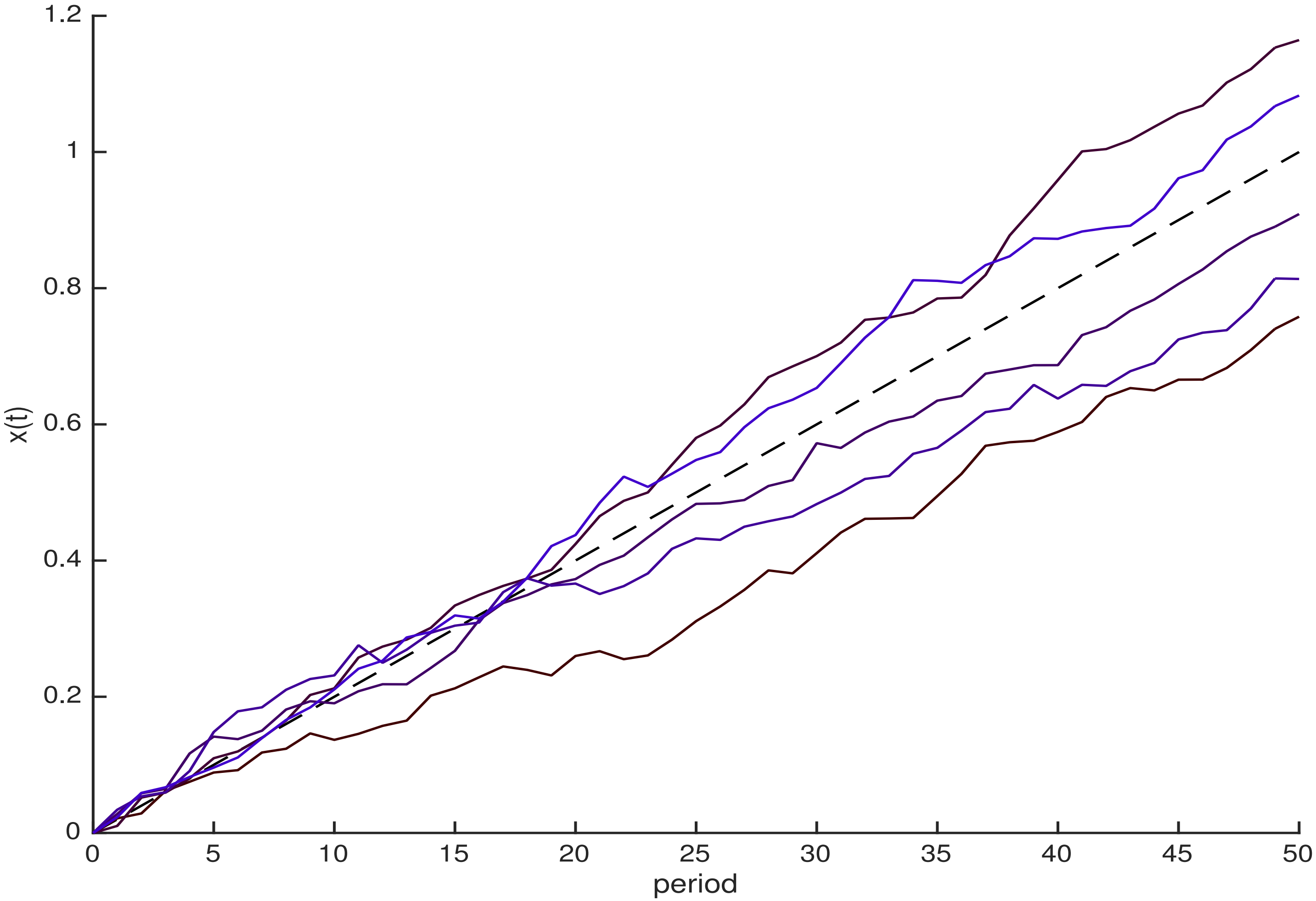
$$x_t = x_0 + \mu t + \sum_{i=1}^{t} \varepsilon_i$$

so that

$$x_t \sim N\left(x_0 + \mu t\,,\ \sigma_\varepsilon^2 t\right)$$

- Mean and variance both linear in t

$$x_{t+1} = \mu + x_t + \varepsilon_{t+1}$$



$x_0 = 0$, $\mu = 0.02$ and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$ with $\sigma_\varepsilon = 0.015$.

Terminology

- The random walk has a 'unit root' $\phi = 1$
- Nonstationary in levels x_t but stationary in first differences Δx_t
- A stochastic process is said to be *integrated of order* d or $I(d)$ if it takes d differences to make the process stationary
- So here x_t is $I(1)$ and Δx_t is $I(0)$

Another example

- Now consider the following AR(2) process

$$x_{t+1} = (1-\phi)\mu + (1+\phi)x_t - \phi x_{t-1} + \varepsilon_{t+1}, \qquad 0 < \phi < 1$$

- Is this process stationary? What the roots of this process?

- Equivalent to system of the form

$$\begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix} = \ldots + \begin{pmatrix} 1+\phi & -\phi \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} + \ldots$$

- Process is stationary if 'system' has stable dynamics,
  i.e., eigenvalues of coefficient matrix are less than one in magnitude

Eigenvalues

- Eigenvalues of $\boldsymbol{A}$ given by roots of the characteristic polynomial

- For the 2-by-2 case

$$p(\lambda) = \lambda^2 - \text{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A})$$

- For this specific example

$$\text{tr}(\boldsymbol{A}) = 1 + \phi > 1, \qquad \det(\boldsymbol{A}) = \phi \in (0, 1)$$

- Using the quadratic formula, for this example roots evaluate to

$$\lambda_1 = 1, \qquad \lambda_2 = \phi$$

- Hence this is also a 'unit root process' and is nonstationary

AR(1) in differences

- In fact this process is simply a stationary AR(1) in differences

$$\Delta x_{t+1} = (1-\phi)\mu + \phi\Delta x_t + \varepsilon_{t+1}, \qquad 0 < \phi < 1$$

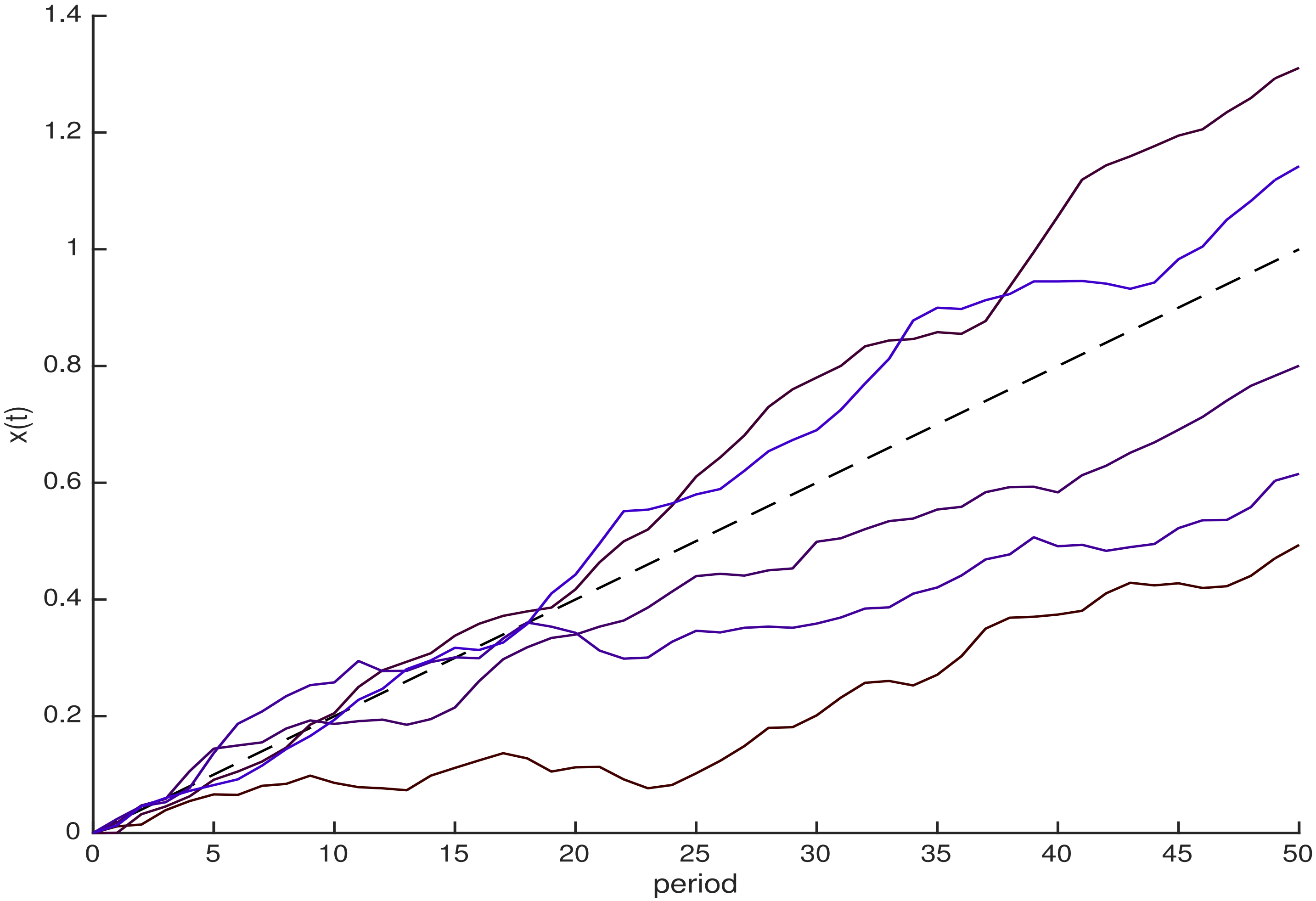
- Since Δx_t is $I(0)$ the process in levels is $I(1)$

- Unlike the random walk, this process implies time-variation in expected growth

$$\mathbb{E}_t[\Delta x_{t+1}] = (1-\phi)\mu + \phi\Delta x_t$$

with reversion to long-run average growth μ governed by ϕ

$$\Delta x_{t+1} = (1-\phi)\mu + \phi \Delta x_t + \varepsilon_{t+1}$$



$x_0 = 0$, $\mu = 0.02$, $\phi = 0.5$ and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$ with $\sigma_\varepsilon = 0.015$.

Cointegration

- A scalar x_t is $I(1)$ if Δx_t is stationary, $I(0)$
- A vector of $I(1)$ variables $\boldsymbol{x}_t$ is *cointegrated* if there is a linear combination $\boldsymbol{a}'\boldsymbol{x}_t$ that is $I(0)$
- For example, if x_t and y_t are both $I(1)$ and $ax_t + by_t$ is $I(0)$ then x_t and y_t are cointegrated
- Implies that there is a *stable long-run relationship* between x_t and y_t even if both nonstationary
- Deviations from $ax_t + by_t$ are mean-reverting, leads to 'error correction' representation

Stochastic growth

- What happens if we use nonstationary processes like this to drive the stochastic growth model?

- Suppose standard aggregate production function, in levels

$$Y_t = F(K_t, A_t L)$$

- Labor-augmenting productivity A_t is stationary in growth rates

$$g_t \equiv \frac{A_t}{A_{t-1}}$$

- Constant employment $L > 0$ for simplicity

Stochastic growth

- Social planner maximizes expected intertemporal utility

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{(C_t/L)^{1-\sigma} - 1}{1-\sigma} \right\}, \qquad 0 < \beta < 1, \quad \sigma > 0$$

  subject to sequence of resource constraints, for each date and state

$$C_t + K_{t+1} = F(K_t, A_t L) + (1-\delta)K_t, \qquad 0 < \delta < 1$$

- Initial $K_0 > 0$ and stochastic process for productivity $\{A_t\}$ given

- Isoelastic utility needed for balanced growth

Intensive form

- In efficiency units

$$y_t \equiv \frac{Y_t}{A_t L}, \qquad c_t \equiv \frac{C_t}{A_t L}, \qquad k_t \equiv \frac{K_t}{A_{t-1} L}$$

- Note capital K_t divided by lagged productivity A_{t-1}

- This implies that detrended k_t remains '*predetermined*'

Detrending

- Resource constraint

$$C_t + K_{t+1} = F(K_t, A_t L) + (1-\delta)K_t$$

- Dividing through by A_t and using $g_t = A_t / A_{t-1}$ gives

$$c_t + k_{t+1} = f(\, k_t \,/\, g_t \,) + (1-\delta)(\, k_t \,/\, g_t \,)$$

where $f(\cdot)$ denotes the intensive form of the production function

- Period utility

$$\frac{(C_t/L)^{1-\sigma} - 1}{1-\sigma} = \frac{(c_t A_t)^{1-\sigma} - 1}{1-\sigma}$$

If log utility, $\sigma \to 1$, period utility is separable in c_t and A_t

Social planner's problem

- Lagrangian with stochastic multiplier $\lambda_t \geq 0$ for each constraint

$$\mathcal{L} = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{(c_t A_t)^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \big[f(k_t/g_t) + (1-\delta)(k_t/g_t) - c_t - k_{t+1} \big] \right\}$$

- Some key first order conditions

$$c_t : \qquad \beta^t c_t^{-\sigma} A_t^{1-\sigma} - \lambda_t = 0$$

$$k_{t+1} : \quad -\lambda_t + \mathbb{E}_t \left\{ \lambda_{t+1} \big[f'(k_{t+1}/g_{t+1})/g_{t+1} + (1-\delta)/g_{t+1} \big] \right\} = 0$$

$$\lambda_t : \qquad f(k_t/g_t) + (1-\delta)(k_t/g_t) - c_t - k_{t+1} = 0$$

- Although k_{t+1} has a $t+1$ subscript, it is chosen conditional on date t information

Dynamical system

- Gives a system of *stationary* stochastic difference equations

$$c_t^{-\sigma} = \beta \mathbb{E}_t \left\{ c_{t+1}^{-\sigma} g_{t+1}^{1-\sigma} \left[f'(k_{t+1}/g_{t+1})/g_{t+1} + (1-\delta)/g_{t+1} \right] \right\}$$

and

$$c_t + k_{t+1} = f(k_t/g_t) + (1-\delta)(k_t/g_t)$$

given initial $k_0 > 0$ and transversality condition

- Maps stationary exogenous $\{g_t\}$ into stationary endogenous $\{c_t, k_t\}$

- The term $\beta g_{t+1}^{1-\sigma}$ in consumption Euler equation is a *growth-adjusted discount factor*

"Non-stochastic steady state"

- Shut down shocks, set $g_t = \bar{g}$
- Find steady state of associated deterministic model
- Steady state capital $\bar{k}$ solves

$$1 = \beta \bar{g}^{-\sigma} \left[f'(\bar{k}/\bar{g}) + 1 - \delta \right]$$

- Steady state consumption $\bar{c}$ pinned down by resource constraint

$$\bar{c} = f(\bar{k}/\bar{g}) + (1-\delta)(\bar{k}/\bar{g}) - \bar{k}$$

Log-linear solution

- Log-linearize the system around these steady state values

$$\hat{k}_t \equiv \log(k_t/\bar{k}), \qquad \hat{c}_t \equiv \log(c_t/\bar{c}), \qquad \hat{g}_t \equiv \log(g_t/\bar{g})$$

- Stationary solution for detrended endogenous variables

$$\hat{k}_{t+1} = \psi_{kk}\hat{k}_t + \psi_{kg}\hat{g}_t$$

and

$$\hat{c}_t = \psi_{ck}\hat{k}_t + \psi_{cg}\hat{g}_t$$

given exogenous stationary process for $\hat{g}_t$

Nonstationary variables

- Log-level of productivity

$$\log A_t = \log A_0 + \sum_{i=1}^{t} \log g_i$$

- Log-levels consumption per worker, capital per worker etc

$$\begin{aligned}
\log(C_t/L) &= \log c_t + \log A_t \\
\log(K_t/L) &= \log k_t + \log A_t - \log g_t \\
\log(Y_t/L) &= \log y_t + \log A_t
\end{aligned}$$

- Each is the sum of a stationary component (from the solution of the detrended model) and a common nonstationary component

- Share a *common stochastic trend*, namely $\log A_t$

Cointegration in the growth model

- Log levels of consumption per worker, capital per worker, output per worker etc are all $I(1)$ because of productivity

- Log consumption/output ratio, capital/output ratio etc are $I(0)$. Consumption and output cointegrated, as are capital and output

$$\log(C_t/Y_t) = \log c_t - \log y_t$$

$$\log(K_t/Y_t) = \log k_t - \log y_t - \log g_t$$

(everything on right hand side is stationary)

- Stable long-run relationships between C_t, Y_t and between K_t, Y_t. Deviations from these long-run relationships are mean-reverting

Parameterization

- CRRA utility function [already imposed for balanced growth]

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \qquad \sigma > 0$$

- Cobb-Douglas production function

$$f(k) = k^{\alpha}, \qquad 0 < \alpha < 1$$

- AR(1) process for log productivity growth

$$\log g_{t+1} = (1 - \phi) \log \bar{g} + \phi \log g_t + \varepsilon_{t+1}, \qquad 0 < \phi < 1$$

with long-run average growth $\log \bar{g}$ and IID normal innovations

$$\varepsilon_{t+1} \sim N(0, \sigma_{\varepsilon}^2)$$

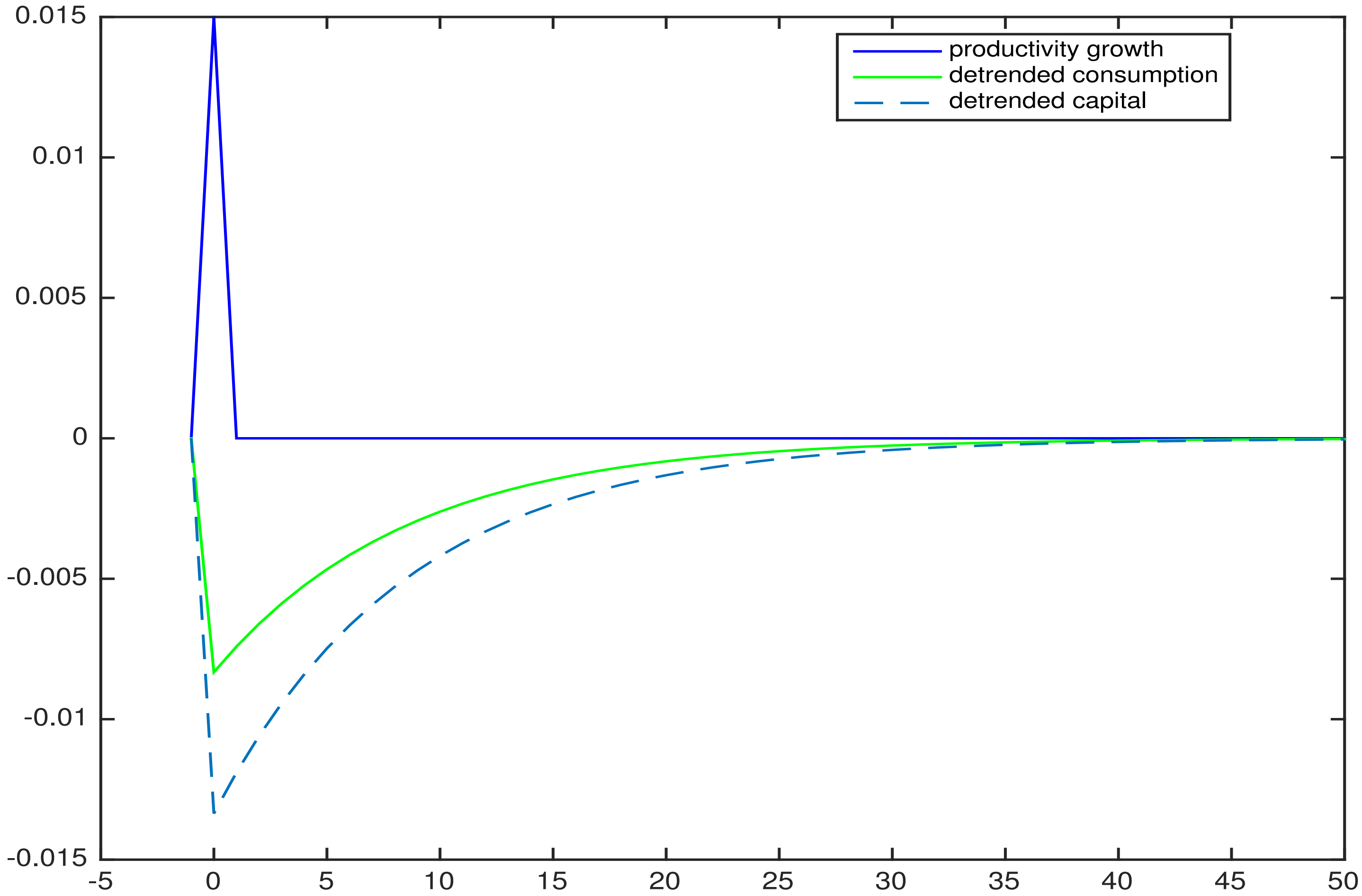
- For examples, let $\sigma = 1$ and $\alpha = 0.3$, $\beta = 0.95$, and $\delta = 0.05$

Example: pure random walk

- Let $\phi = 0$ with $\bar{g} = 1.00$ and $\sigma_\varepsilon = 0.015$
- No growth but shocks have permanent effect on levels
- Dynare gives coefficients

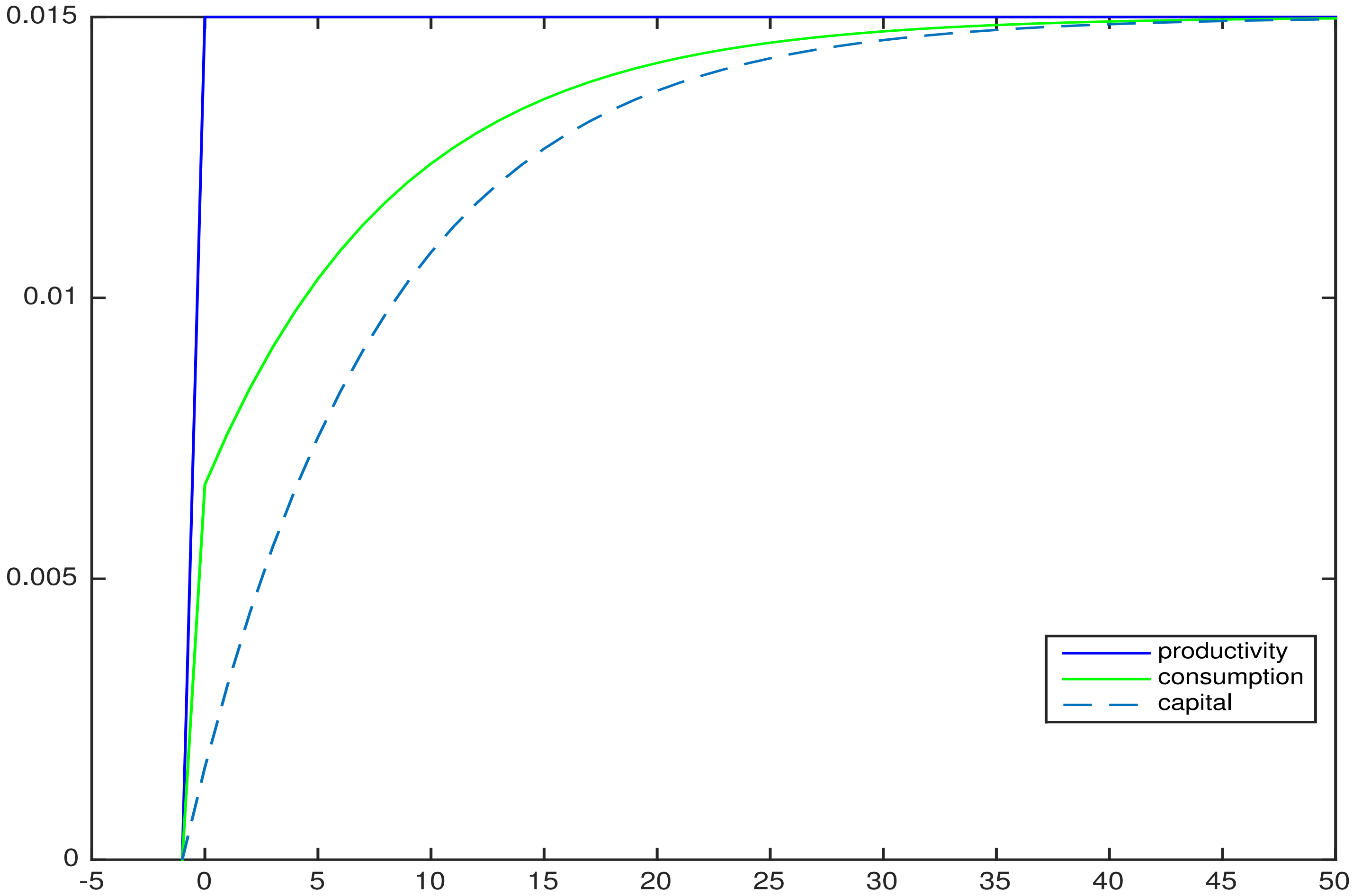
$$\begin{pmatrix} \psi_{kk} & \psi_{kg} \\ \psi_{ck} & \psi_{cg} \end{pmatrix} = \begin{pmatrix} 0.89 & -0.89 \\ 0.55 & -0.55 \end{pmatrix}$$

Detrended variables are mean-reverting



Impulse response functions of detrended variables $\hat{c}_t$ and $\hat{k}_t$ to 1 standard deviation shock when log productivity is a pure random walk.

But shocks have permanent effect on levels



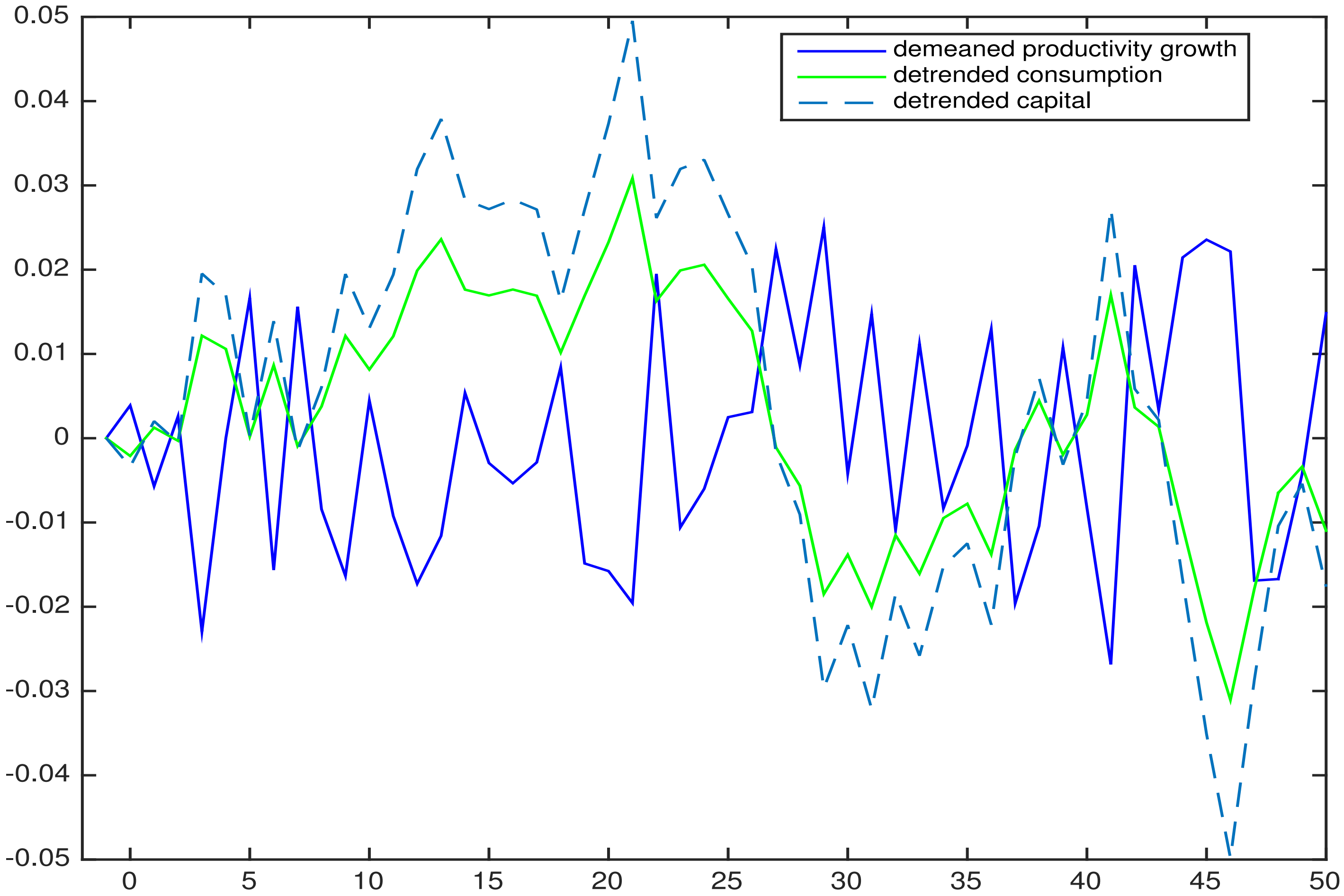
Impulse response functions of levels $\log C_t$ and $\log K_t$ to 1 standard deviation shock when $\log A_t$ is a pure random walk.

Example: random walk with drift

- Let $\phi = 0$ with $\bar{g} = 1.02$ and $\sigma_\varepsilon = 0.015$
- Expected growth 2% no matter what current g_t is
- Dynare now gives coefficients

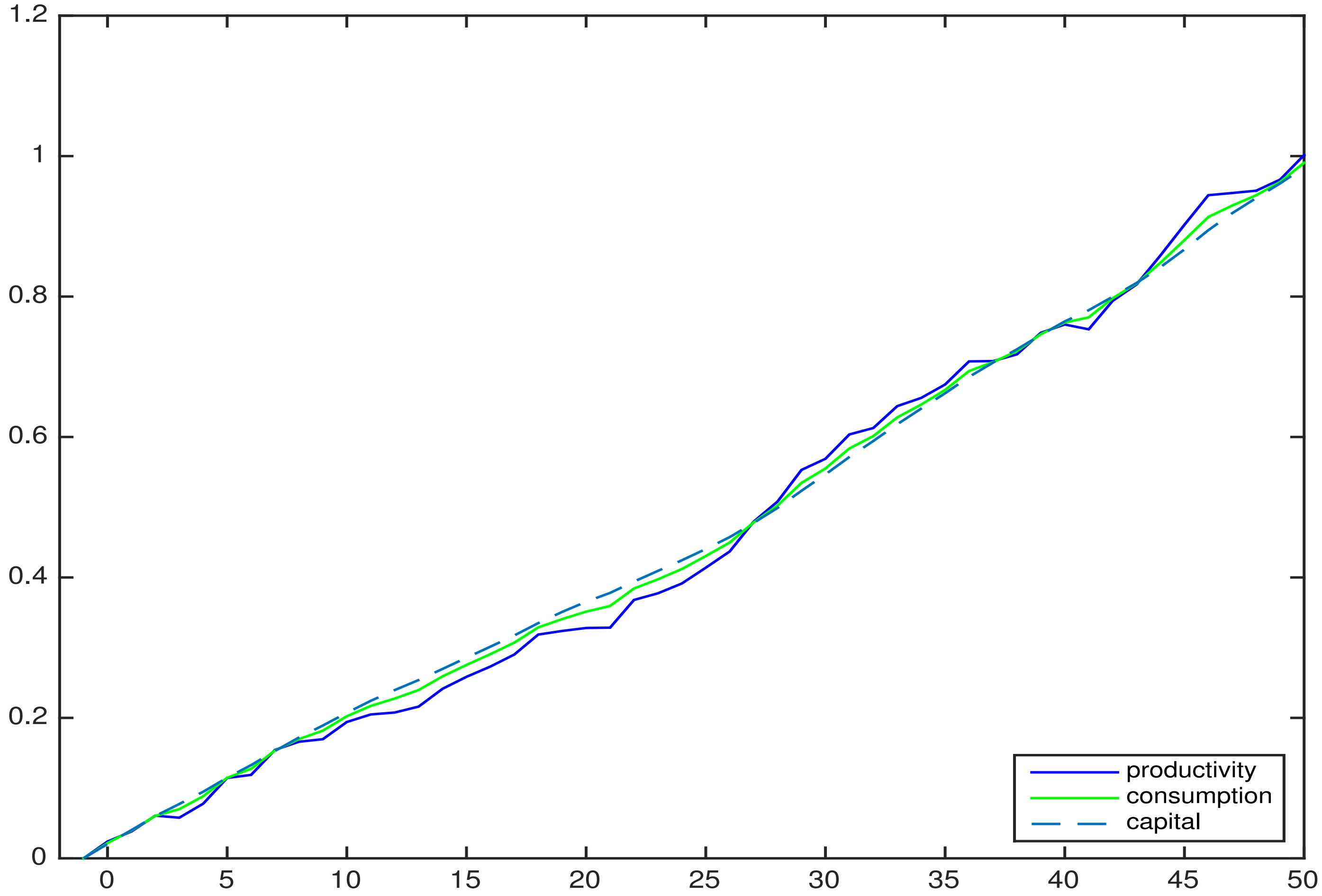
$$\begin{pmatrix} \psi_{kk} & \psi_{kg} \\ \psi_{ck} & \psi_{cg} \end{pmatrix} = \begin{pmatrix} 0.87 & -0.87 \\ 0.54 & -0.54 \end{pmatrix}$$

Stationary variables



Simulated time series for detrended variables $\hat{c}_t$ and $\hat{k}_t$ when log productivity is a random walk with drift.

Nonstationary variables



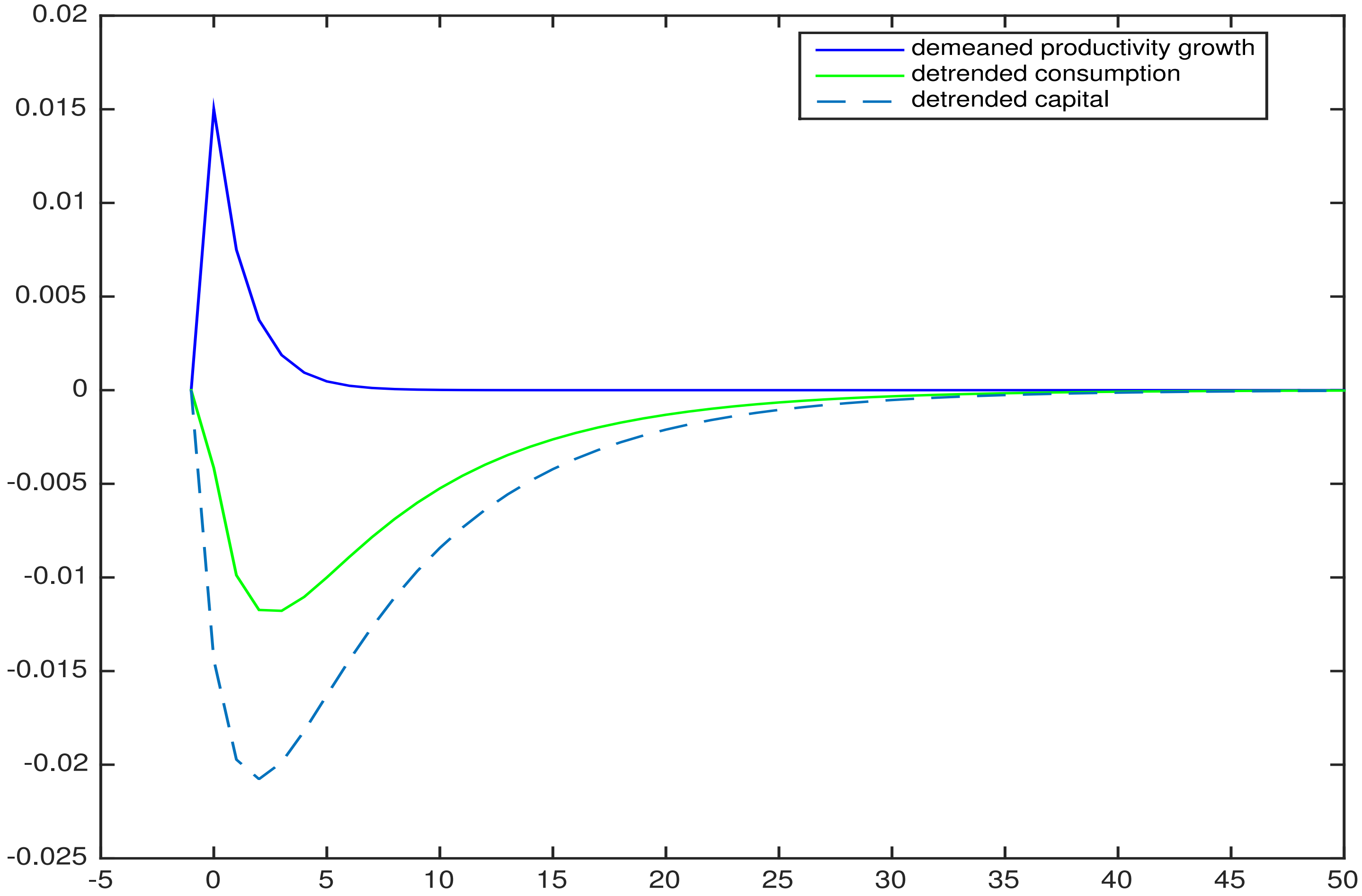
Simulated time series for levels $\log C_t$ and $\log K_t$ when $\log A_t$ is a random walk with drift. Levels nonstationary but ratios $\log(C_t/Y_t)$ and $\log(K_t/Y_t)$ are stationary.

Example: AR(1) in growth rates

- Let $\phi = 0.5$ with $\bar{g} = 1.02$ and $\sigma_\varepsilon = 0.015$
- Expected growth is time-varying, fluctuates around 2%
- Dynare now gives coefficients

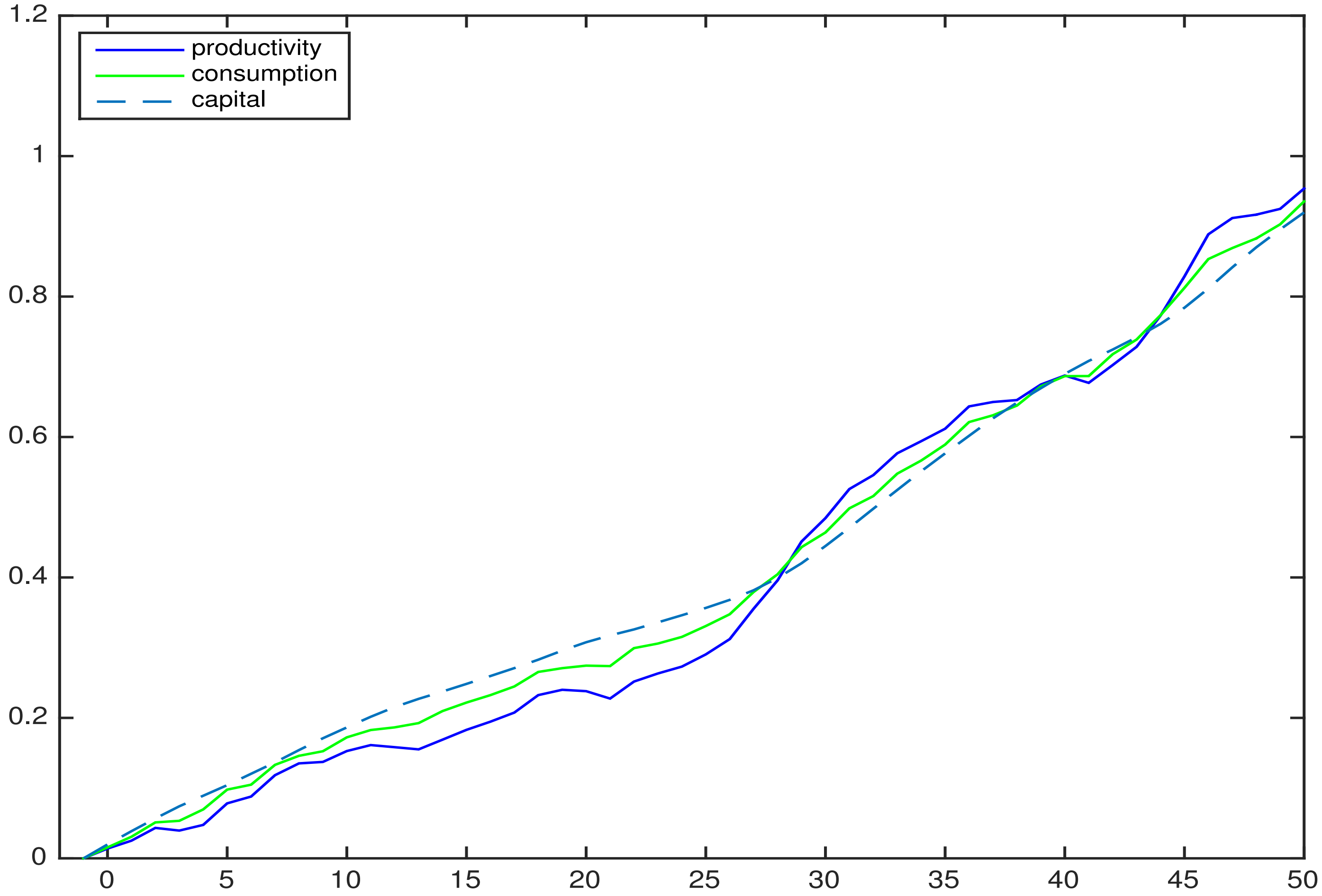
$$\begin{pmatrix} \psi_{kk} & \psi_{kg} \\ \psi_{ck} & \psi_{cg} \end{pmatrix} = \begin{pmatrix} 0.87 & -0.96 \\ 0.54 & -0.28 \end{pmatrix}$$

Stationary variables



Impulse response functions of detrended variables $\hat{c}_t$ and $\hat{k}_t$ to 1 standard deviation shock when log productivity growth is an AR(1).

Nonstationary variables



Simulated time series for levels $\log C_t$ and $\log K_t$ when log productivity growth is an AR(1). Low-frequency fluctuations in stationary ratios more pronounced.

Next class

- Beginning of lectures on monetary economics