Advanced Macroeconomics

Nonlinear dynamics and endogenous risk

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This class

- Nonlinear dynamics and endogenous risk

- Further reading

  - Brunnermeier and Sannikov (2014): A macroeconomic model with a financial sector, *American Economic Review*.

Brunnermeier-Sannikov

- Continuous time $t \geq 0$, aggregate shocks
- Two types of agents, *experts* (entrepreneurs) and *households*
- Differ in three ways

  (i) experts more productive
  (ii) experts less patient
  (iii) experts subject to nonnegativity constraint, impedes risk bearing

- Exogenous interest rate r

Technology

- Experts produce flow output

$$y_t = ak_t, \qquad a > 0$$

with capital driven by aggregate shocks (Brownian motion) and subject to adjustment costs

$$dk_t = (\Phi(\iota_t) - \delta)k_t\,dt + \sigma k_t dz_t$$

where $\iota = i_t/k_t$ denotes investment per unit capital

- Households less productive, produce flow output

$$\underline{y}_t = \underline{a}\,\underline{k}_t, \qquad 0 < \underline{a} < a$$

with

$$d\underline{k}_t = (\Phi(\underline{\iota}_t) - \underline{\delta})\underline{k}_t\,dt + \sigma \underline{k}_t dz_t, \qquad \underline{\delta} > \delta$$

First best

- In frictionless economy

  - experts would manage all capital
  - consume lifetime wealth at $t = 0$ (since impatient)
  - issue equity to households
  - *first-best price of capital*, given by present value

$$\overline{q} = \max_{\iota} \left[\frac{a - \iota}{r - (\Phi(\iota) - \delta)} \right]$$

- But if experts cannot issue equity, need to maintain positive net worth as buffer against risk (given nonnegative consumption)

- If net worth drops to zero, cannot hold any capital and price of capital drops to *liquidation value*

$$\underline{q} = \max_{\iota} \left[\frac{\underline{a} - \iota}{r - (\Phi(\iota) - \underline{\delta})} \right] < \overline{q}$$

Market structure

- By assumption, experts must retain all equity and can issue only noncontingent debt

- If expert net worth ever reaches zero, can no longer absorb risk. Sell all capital and consume nothing from that point on

- Market price of capital driven by aggregate shocks dz_t

$$dq_t = \mu_t^q q_t \, dt + \sigma_t^q q_t \, dz_t$$

  with drift μ_t^q and volatility σ_t^q to be determined in equilibrium

- Bounded by $\underline{q}, \overline{q}$

Household problem

- Choose consumption $d\underline{c}_t$ and share of wealth $\underline{x}_t$ in capital to max

$$\mathbb{E}_0 \left\{ \int_0^\infty e^{-rt}\, d\underline{c}_t \right\}, \qquad r > 0$$

subject to flow constraint for net worth $\underline{n}_t$

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t\, d\underline{r}_t^k + (1 - \underline{x}_t) r\, dt - \frac{d\underline{c}_t}{\underline{n}_t}$$

and nonnegativity constraints

$$\underline{n}_t \geq 0, \quad \underline{x}_t \geq 0$$

- Household consumption can be negative (e.g., disutility from labor)

- Return on capital $d\underline{r}_t^k$ driven by aggregate shocks dz_t

Household problem

- Let ψ_t denote fraction of aggregate capital K_t held by experts
- Then $1 - \psi_t$ is fraction of aggregate capital held by households
- Optimality condition for households

$$\mathbb{E}_t \left\{ d\underline{r}_t^k \right\} \leq r\, dt$$

with equality whenever $1 - \psi_t > 0$

- Not constrained, so must earn r from holding capital if they do so

Expert problem

- Choose consumption dc_t and share of wealth x_t in capital to max

$$\mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \, dc_t \right\}, \qquad \rho > r$$

subject to flow constraint for net worth n_t

$$\frac{dn_t}{n_t} = x_t \, dr_t^k + (1 - x_t) r \, dt - \frac{dc_t}{n_t}$$

and nonnegativity constraints

$$dc_t \geq 0, \quad n_t \geq 0, \quad x_t \geq 0$$

- Anticipate that in general $x_t > 1$, i.e., experts *levered*

- Return on capital dr_t^k driven by aggregate shocks dz_t

Expert problem

- Let θ_t denote the marginal value of expert net worth. Can write

$$\theta_t n_t \equiv \mathbb{E}_t \left\{ \int_0^\infty e^{-\rho(t-s)} \, dc_s \right\}$$

maximised subject to the constraints above

- Solves

$$\frac{d\theta_t}{\theta_t} = (\rho - r) \, dt + \sigma_t^\theta dz_t$$

with endogenous *risk premium*

$$-\sigma_t^\theta (\sigma + \sigma_t^q) \, dt \geq \mathbb{E}_t \left\{ dr_t^k \right\} - r \, dt$$

with equality if $x_t > 0$

Wealth distribution dynamics

- Let N_t denote aggregate expert net worth. Then $q_t K_t - N_t$ is aggregate household wealth

- Let $\eta_t \equiv N_t/(q_t K_t)$ denote *expert share of aggregate wealth*. The key state variable for this model

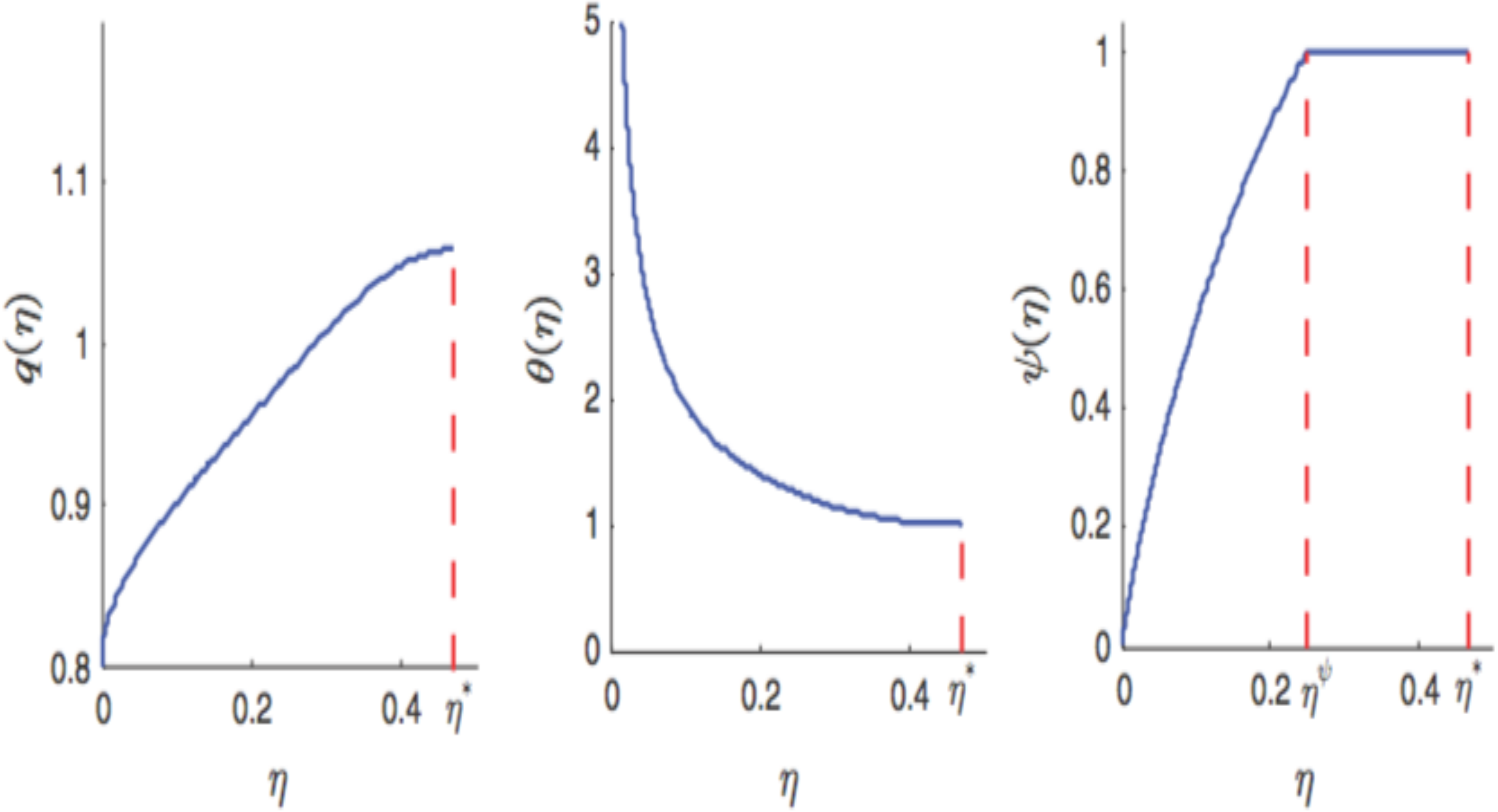
- Equilibrium summarized by three functions to be determined

  (i) $q_t = q(\eta_t)$, the price of capital
  (ii) $\theta_t = \theta(\eta_t)$, the marginal value of expert net worth
  (iii) $\psi_t = \psi(\eta_t)$, fraction of capital stock held by experts

- Brunnermeier and Sannikov solve implied system of differential equations numerically [see paper for details]

- Experts more constrained when η_t falls, reduces $q(\eta_t)$ and $\psi(\eta_t)$

Equilibrium $q(\eta), \theta(\eta), \psi(\eta)$



As expert wealth share η increases, price of capital $q(\eta)$ increases and marginal value of expert wealth $\theta(\eta)$ falls (hence precautionary savings motive). Experts hold all capital when $\eta \in [\eta^{\psi}, \eta^*]$. For $\eta \geq \eta^*$, $\theta(\eta) = 1$. For such η, good shocks consumed away. For bad shocks, $\eta < \eta^*$, experts do not consume, and system drifts back to η^*.

Instability and endogenous risk

- Price of capital subject to endogenous risk σ_t^q
- Amount of endogenous risk varies with state η
  - low risk near stochastic steady state η^*
  - high risk near critical point η^ψ (boundary for $\psi(\eta) = 1$)
  - stream of bad shocks can push η into high risk region
  - critical point η^ψ where experts start selling capital to households
- Standard models look at *local dynamics*
  (i.e., log-linear approximations around deterministic steady state)
- This may miss important features of the *global dynamics*

Endogenous risk

- Depends on sensitivity of price of capital $q(\eta)$ to η

$$\sigma_t^q = \frac{q'(\eta)\eta}{q(\eta)}\sigma_t^\eta$$

where σ_t^η is the volatility of the expert wealth share, given by

$$\sigma_t^\eta = \frac{\left(\frac{\psi(\eta)}{\eta} - 1\right)}{1 - \left(\frac{\psi(\eta)}{\eta} - 1\right)\frac{q'(\eta)\eta}{q(\eta)}}\sigma$$

where $\psi(\eta)/\eta$ is the expert leverage ratio

Amplification: intuition

- Amount of amplification depends on

  (i) extent of expert leverage $\psi(\eta)/\eta$
  (ii) sensitivity of capital price $q(\eta)$ to η, feedback to net worth

- Direct effect of shock that reduces aggregate capital

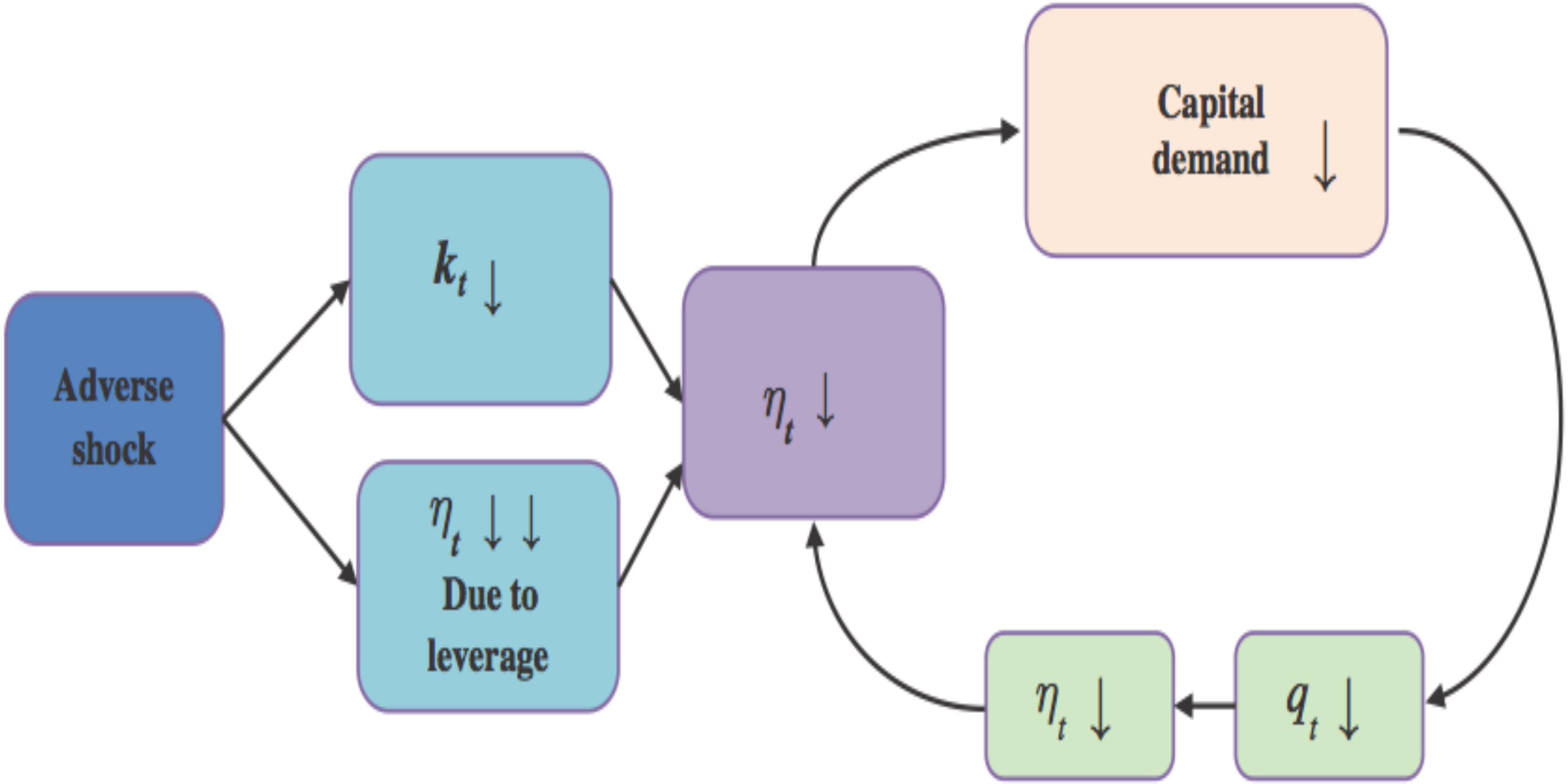
$$\frac{\psi(\eta)}{\eta} - 1 \qquad \text{percent fall in expert wealth share } \eta_t$$

- Price response

$$\phi \equiv \frac{q'(\eta)\eta}{q(\eta)}\left(\frac{\psi(\eta)}{\eta} - 1\right) \qquad \text{percent fall in price of capital } q(\eta_t)$$

- Multiplier-like effect: wealth share falls by further $\left(\frac{\psi(\eta)}{\eta} - 1\right)\phi$, further ϕ^2 price response etc etc

Adverse feedback loop



Adverse shock reduces expert wealth share η_t both directly and because a falling wealth share reduces expert demand for capital which reduces price of capital q_t which further reduces expert wealth share.

Amplification: intuition

- Cumulative amplification (supposing $0 < \phi < 1$)

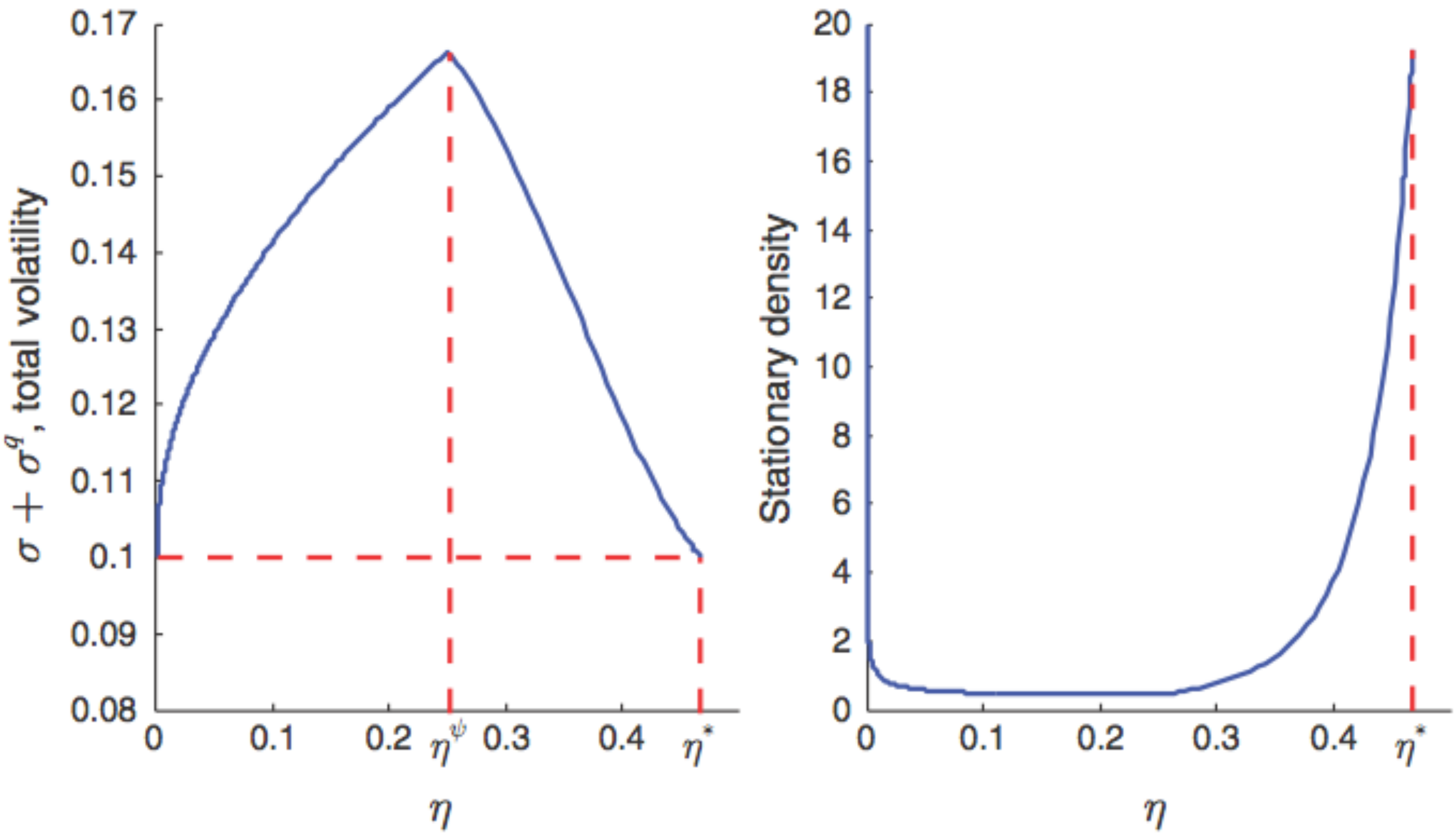
$$\frac{d\eta_t}{\eta_t} = \frac{1}{1-\phi}\left(\frac{\psi(\eta)}{\eta} - 1\right) = \frac{\left(\frac{\psi(\eta)}{\eta} - 1\right)}{1 - \left(\frac{\psi(\eta)}{\eta} - 1\right)\frac{q'(\eta)\eta}{q(\eta)}}$$

and

$$\frac{dq_t}{q_t} = \left(\frac{q'(\eta)\eta}{q(\eta)}\right)\frac{d\eta_t}{\eta_t}$$

- Near η^* have $q'(\eta^*) = 0$, i.e., no price amplification near η^*, only leverage effect. But away from η^* have $q'(\eta)$ relatively high and additional price amplification channel

Bimodal stationary distribution of η_t



Stationary distribution of expert wealth share η_t is *bimodal*, with one peak at stochastic steady state η^* and another at zero. Total volatility $\sigma + \sigma_t^q$ peaks at η^ψ. Total volatility is low at both η^* (where $q_t \approx \overline{q}$) and at zero (where $q_t \approx \underline{q}$). Density between peaks is relatively low, system travels relatively quickly between extremes.

Volatility paradox

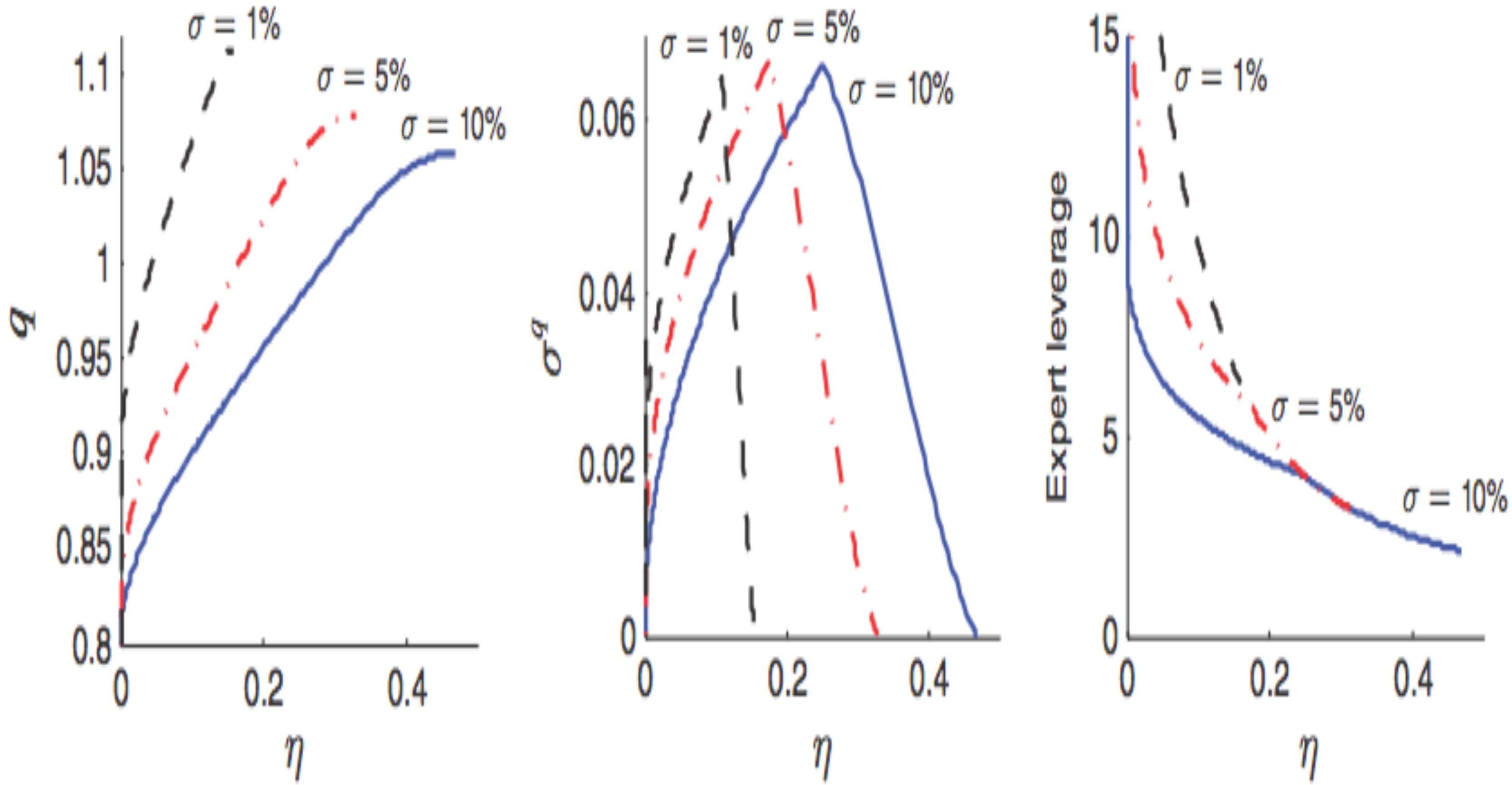
- Does endogenous risk σ_t^q go to zero as exogenous risk $\sigma \to 0$?
- Perhaps surprisingly, the answer is *no*.
- Intuitively, when σ is low, experts more willing to lever up
- Implies price of capital more sensitive to η_t, hence more amplification and hence endogenous risk remains even when σ low

Stochastic vs. deterministic steady states

- η^* is the *stochastic steady state* expert wealth share (i.e., the point of global attraction of the system)
  - a function of σ
  - internalizes the effects of endogenous risk σ_t^q
- Let η^0 denote the *deterministic steady state* expert wealth share (i.e., the share in the complete absence of shocks)
  - there is a discontinuity

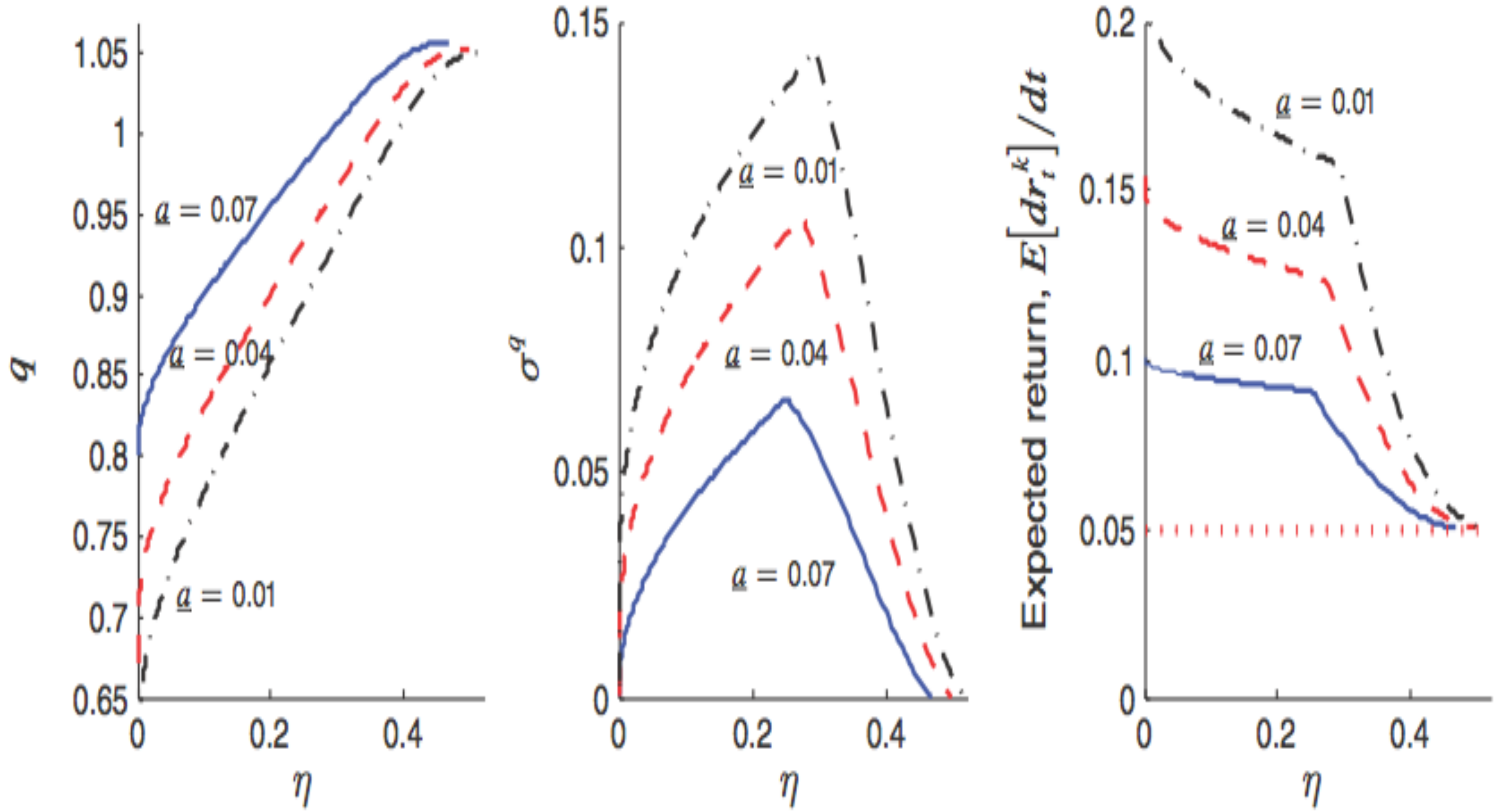
$$\lim_{\sigma \to 0} \eta^* \neq \eta^0$$

Volatility paradox



Lower exogenous risk σ encourages more leverage. Price of capital $q(\eta_t)$ more sensitive to η_t. Peak endogenous risk σ_t^q just as high (though location of peak and hence location of crisis region shifts).

Market illiquidity and endogenous risk



Market liquidity — i.e., the *gap* between $\underline{q}$ and $\overline{q}$ — determines the extent of endogenous risk. When $\underline{a}$ is lower there is a bigger gap between $\underline{q}$ and $\overline{q}$ and endogenous risk and risk premia are greater.