316-632 International Monetary Economics

Problem Set #2 *Selected answers*

Question 1. First of all, note that we **cannot** simply combine the static resource constraint and the law of motion for capital to get an expression like

$$"c_t + k_{t+1} = z_t F(k_t, n_t) + (1-\delta)k_t - \phi\left(\frac{i_t}{k_t}\right)k_t"$$

This manipulation does not entirely eliminate the investment choice from the problem. In fact, it gives us an ill-posed problem where investment has a cost but no benefit. Clearly, this does not reflect the economics of the problem. Instead, we have to keep separate account of the resource constraint, $c_t + i_t = z_t F(k_t, n_t)$ and the law of motion for capital, $k_{t+1} = (1-\delta)k_t + i_t - \phi\left(\frac{i_t}{k_t}\right)k_t$. One way to do this is to write a Lagrangian of the form

$$L = \mathsf{E}_0\left\{\sum_{t=0}^{\infty}\beta^t\{U[z_tF(k_t,n_t)-i_t]+V(1-n_t)\}+\sum_{t=0}^{\infty}\lambda_t\beta^t\left[(1-\delta)k_t+i_t-\phi\left(\frac{i_t}{k_t}\right)k_t-k_{t+1}\right]\right\}$$

with multipliers $\lambda_t\beta^t \geq 0$. The key first order conditions for this problem include, by choice of n_t,

$$V'(\ell_t) = U'(c_t)z_tF_n(k_t,n_t)$$

So we get the standard relationship between the marginal rate of substitution of labor for consumption and the marginal product of labor, namely

$$\frac{V'(\ell_t)}{U'(c_t)} = z_tF_n(k_t,n_t)$$

We also have, by choice of i_t,

$$U'(c_t) = \lambda_t\left[1-\phi'\left(\frac{i_t}{k_t}\right)\right]$$

If there were no adjustment costs, $\phi = \phi' = 0$, then $\lambda_t = \eta_t$ and we would have the standard consumption Euler equation for capital accumulation. With adjustment costs, investment and installed capital are **not perfect substitutes** and we have to translate between their shadow values. We also have the Euler equation for capital accumulation k_{t+1},

$$\lambda_t = \mathsf{E}_t\left\{\beta^{t+1}U'(c_{t+1})z_{t+1}F_k(k_{t+1},n_{t+1})+\lambda_{t+1}\left[(1-\delta)-\phi\left(\frac{i_{t+1}}{k_{t+1}}\right)+\phi'\left(\frac{i_{t+1}}{k_{t+1}}\right)\frac{i_{t+1}}{k_{t+1}}\right]\right\}$$

and on eliminating the marginal utility of consumption

$$\lambda_t = \mathsf{E}_t\left\{\lambda_{t+1}\left[1-\phi'\left(\frac{i_{t+1}}{k_{t+1}}\right)\right]z_{t+1}F_k(k_{t+1},n_{t+1})+\lambda_{t+1}\left[(1-\delta)-\phi\left(\frac{i_{t+1}}{k_{t+1}}\right)+\phi'\left(\frac{i_{t+1}}{k_{t+1}}\right)\frac{i_{t+1}}{k_{t+1}}\right]\right\}$$

This is a standard asset pricing formula, namely

$$1 = \mathsf{E}_t\left\{\frac{\lambda_{t+1}}{\lambda_t}R_{t+1}\right\}$$

where the gross return on installed capital is

$$R_{t+1} \equiv 1 - \delta - \phi\left(\frac{i_{t+1}}{k_{t+1}}\right) + \phi'\left(\frac{i_{t+1}}{k_{t+1}}\right)\frac{i_{t+1}}{k_{t+1}} + \left[1 - \phi'\left(\frac{i_{t+1}}{k_{t+1}}\right)\right] z_{t+1}F_k(k_{t+1}, n_{t+1})$$

Question 2. At a non-stochastic steady state, we have $z_t = z_{t+1} = \bar{z} = 1$ and $k_{t+1} = k_t = \bar{k}$. The law of motion for capital therefore implies that steady state investment and capital are related by

$$\bar{k} = (1-\delta)\bar{k} + \bar{\imath} - \phi\left(\frac{\bar{\imath}}{\bar{k}}\right)\bar{k}$$

or

$$\frac{\bar{\imath}}{\bar{k}} = \delta + \phi\left(\frac{\bar{\imath}}{\bar{k}}\right)$$

Given that ϕ is a strictly increasing strictly convex function with the properties $\phi(\delta) = \phi'(\delta) = 0$, the only solution is $\bar{\imath} = \delta\bar{k}$. Therefore, the steady state gross rate of return on capital is

$$\begin{aligned}\bar{R} &= 1 - \delta - \phi(\delta) + \phi'(\delta)\,\delta + [1 - \phi'(\delta)]\,F_k(\bar{k}, \bar{n}) \\ &= 1 - \delta + F_k(\bar{k}, \bar{n})\end{aligned}$$

The marginal utility of consumption satisfies

$$U'(\bar{c}) = \bar{\lambda}$$

so at a non-stochastic steady state we also have

$$\begin{aligned}1 &= \beta\bar{R} \\ &= \beta\left[1 - \delta + F_k(\bar{k}, \bar{n})\right] \qquad (1)\end{aligned}$$

We also have the labor supply condition

$$\frac{V'(1-\bar{n})}{U'(\bar{c})} = F_n(\bar{k}, \bar{n}) \qquad (2)$$

and the resource constraint

$$\bar{c} + \delta\bar{k} = F(\bar{k}, \bar{n}) \qquad (3)$$

These constitute a system of three non-linear equations in the three unknowns, $(\bar{c}, \bar{n}, \bar{k})$. With the assumed functional forms, they can be simplified to

$$\begin{aligned}1 &= \beta\left[1 - \delta + \theta\left(\frac{\bar{k}}{\bar{n}}\right)^{\theta-1}\right] \\ \frac{\bar{c}}{1-\bar{n}} &= (1-\theta)\left(\frac{\bar{k}}{\bar{n}}\right)^{\theta} \\ \bar{c} + \delta\bar{k} &= \bar{k}^{\theta}\bar{n}^{1-\theta}\end{aligned}$$

To solve this system "by-hand", we begin by inverting the steady state Euler equation to get the capital labor ratio

$$\frac{\bar{k}}{\bar{n}} = \left(\frac{\theta\beta}{1-\beta+\delta\beta}\right)^{\frac{1}{1-\theta}}$$

Hence the consumption/leisure ratio is just

$$\frac{\bar{c}}{1-\bar{n}} = (1-\theta)\left(\frac{\theta\beta}{1-\beta+\delta\beta}\right)^{\frac{\theta}{1-\theta}} \equiv \Gamma$$

To complete the solution, we need to solve for consumption and labor separately. To do this, write the resource constraint as

$$\frac{\bar{c}}{\bar{n}} + \delta\frac{\bar{k}}{\bar{n}} = \left(\frac{\bar{k}}{\bar{n}}\right)^{\theta}$$

or

$$\frac{\bar{c}}{\bar{n}} = \left(\frac{\theta\beta}{1-\beta+\delta\beta}\right)^{\frac{\theta}{1-\theta}} - \delta\left(\frac{\theta\beta}{1-\beta+\delta\beta}\right)^{\frac{1}{1-\theta}} \equiv \Lambda$$

Hence

$$\begin{aligned} \bar{c} &= \Gamma(1-\bar{n}) \\ \bar{c} &= \Lambda\bar{n} \end{aligned}$$

constitutes a system of two linear equations in two unknowns. The solutions are

$$\begin{aligned} \bar{n} &= \frac{\Gamma}{\Lambda+\Gamma} \\ \bar{c} &= \frac{\Lambda\Gamma}{\Lambda+\Gamma} \end{aligned}$$

With the solution for steady state employment in hand, we can back out capital from our previous calculation, namely

$$\bar{k} = \left(\frac{\theta\beta}{1-\beta+\delta\beta}\right)^{\frac{1}{1-\theta}} \bar{n}$$

Using the given parameters, I obtain

$$\begin{aligned} \bar{n} &= 0.4763 \\ \bar{c} &= 0.8879 \\ \bar{k} &= 7.9399 \end{aligned}$$

(see the attached Matlab code for more details).

Question 3. The log-linear versions of most of the equations are straightforward. First, labor supply is

$$\frac{V'(\ell_t)}{U'(c_t)} = z_t F_n(k_t, n_t)$$

and approximately

$$\frac{V''(\bar{\ell})\bar{\ell}}{V'(\bar{\ell})}\hat{\ell}_t - \frac{U''(\bar{c})\bar{c}}{U'(\bar{c})}\hat{c}_t = \hat{z}_t + \frac{F_{nk}(\bar{k},\bar{n})\bar{k}}{F_n(\bar{k},\bar{n})}\hat{k}_t + \frac{F_{nn}(\bar{k},\bar{n})\bar{n}}{F_n(\bar{k},\bar{n})}\hat{n}_t$$

With our assumed functional forms, the log-linear labor supply condition can be written

$$-\hat{\ell}_t + \hat{c}_t = \hat{z}_t + \theta \hat{k}_t - \theta \hat{n}_t$$

and on using

$$(1-\bar{n})\hat{\ell}_t + \bar{n}\hat{n}_t = 0$$

we can eliminate leisure to get

$$0 = \hat{z}_t + \theta \hat{k}_t - \left(\theta + \frac{\bar{n}}{1-\bar{n}}\right)\hat{n}_t - \hat{c}_t \tag{4}$$

The key equation to log-linearize is the gross marginal product of capital, which I will write as

$$\begin{aligned} R_{t+1} &\equiv 1-\delta-\phi\left(x_{t+1}\right)+\phi'\left(x_{t+1}\right)x_{t+1}+\left[1-\phi'\left(x_{t+1}\right)\right]z_{t+1}F_k(k_{t+1},n_{t+1}) \\ x_{t+1} &\equiv \frac{i_{t+1}}{k_{t+1}} \end{aligned}$$

Clearly the investment/capital ratio satisfies

$$\begin{aligned} \hat{x}_{t+1} &= \hat{\imath}_{t+1} - \hat{k}_{t+1} \\ \bar{x} &= \delta \end{aligned}$$

Log linearizing the return

$$\hat{R}_{t+1} = \beta[\phi''(\delta)\,\delta - \phi''(\delta)\,\bar{r}]\bar{x}\hat{x}_{t+1} + \beta\bar{r}\hat{z}_{t+1} + F_{kk}(\bar{k},\bar{n})\bar{k}\hat{k}_{t+1} + F_{kn}(\bar{k},\bar{n})\bar{n}\hat{n}_{t+1}$$

(this uses the steady state relationships $\bar{R} = 1-\delta+F_k(\bar{k},\bar{n}) = \beta^{-1}$, $F_k(\bar{k},\bar{n}) = \bar{r}$ and $\phi'(\delta) = 0$). Now using the functional form for ϕ, $\phi''(x) = 1$ all x so $\phi''(\delta) = 1$. Also,

$$\begin{aligned} F_{kk}(\bar{k},\bar{n}) &= -(1-\theta)\frac{F_k(\bar{k},\bar{n})}{\bar{k}} \\ F_{kn}(\bar{k},\bar{n}) &= (1-\theta)\frac{F_k(\bar{k},\bar{n})}{\bar{n}} \end{aligned}$$

So we can write

$$\hat{R}_{t+1} = \beta(\delta-\bar{r})\delta\hat{x}_{t+1} + \beta\bar{r}\hat{z}_{t+1} - (1-\theta)\beta\bar{r}(\hat{k}_{t+1}-\hat{n}_{t+1}) \tag{5}$$

Similarly, the resource constraint and law of motion for capital are

$$\begin{aligned} \bar{c}c_t + \bar{\imath}i_t &= \hat{z}_t + F_k(\bar{k},\bar{n})\bar{k}\hat{k}_t + F_n(\bar{k},\bar{n})\bar{n}\hat{n}_t \\ &= \hat{z}_t + \theta\bar{y}\hat{k}_t + (1-\theta)\bar{y}\hat{n}_t \end{aligned} \tag{6}$$

and

$$\begin{aligned} \bar{k}\hat{k}_{t+1} &= [(1-\delta)-\phi(\delta)]\bar{k}\hat{k}_t + \bar{\imath}i_t - \phi'(\delta)\,\bar{k}\hat{x}_t \\ &= (1-\delta)\bar{k}\hat{k}_t + \bar{\imath}i_t \end{aligned} \tag{7}$$

So to a first order approximation, the costs of adjustment do not affect the capital accumulation equation.

Finally, we have the consumption Euler equation

$$0 = \mathsf{E}_t\{\hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{R}_{t+1}\}$$

where

$$\frac{U''(\bar{c})\bar{c}}{U'(\bar{c})}\hat{c}_t = \hat{\lambda}_t - \delta\hat{x}_t$$

and making use of the functional form for $U(c)$, we have

$$-\hat{c}_t = \hat{\lambda}_t - \delta\hat{x}_t$$

I define

$$\begin{aligned} X_t &\equiv \hat{k}_{t+1} \\ Y_t &\equiv \begin{pmatrix} \hat{c}_t \\ \hat{n}_t \\ \hat{x}_t \\ \hat{\imath}_t \\ \hat{R}_t \\ \hat{\lambda}_t \end{pmatrix} \\ Z_t &\equiv \hat{z}_t \end{aligned}$$

And so my system of static equations is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\bar{k} \\ 0 \end{pmatrix}}_{A} \hat{k}_{t+1} + \underbrace{\begin{pmatrix} \theta \\ 1 \\ -(1-\theta)\beta\bar{r} \\ \theta\bar{y} \\ (1-\delta)\bar{k} \\ 0 \end{pmatrix}}_{B} \hat{k}_t$$
$$+ \underbrace{\begin{pmatrix} -1 & -\left(\theta + \frac{\bar{n}}{1-\bar{n}}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & (1-\theta)\beta\bar{r} & (\delta-\bar{r})\beta\delta & 0 & -1 & 0 \\ -\bar{c} & (1-\theta)\bar{y} & 0 & -\bar{\imath} & 0 & 0 \\ 0 & 0 & 0 & \bar{\imath} & 0 & 0 \\ 1 & 0 & -\delta & 0 & 0 & 1 \end{pmatrix}}_{C} \begin{pmatrix} \hat{c}_t \\ \hat{n}_t \\ \hat{x}_t \\ \hat{\imath}_t \\ \hat{R}_t \\ \hat{\lambda}_t \end{pmatrix} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ \beta\bar{r} \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{D} \hat{z}_t$$

There is only one forward-looking equation,

$$0 = \mathsf{E}_t\{\hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{R}_{t+1}\}$$

or

$$0 = \mathsf{E}_t \left\{ (0)\,\hat{k}_{t+2} + (0)\,\hat{k}_{t+1} + (0)\,\hat{k}_t + \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \hat{c}_{t+1} \\ \hat{n}_{t+1} \\ \hat{x}_{t+1} \\ \hat{\imath}_{t+1} \\ \hat{R}_{t+1} \\ \hat{\lambda}_{t+1} \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{n}_t \\ \hat{x}_t \\ \hat{\imath}_t \\ \hat{R}_t \\ \hat{\lambda}_t \end{pmatrix} + (0)\,\hat{z}_{t+1} + (0)\,\hat{z}_t \right\}$$

Question 4. Using Harald Uhlig's toolkit (see the attached code for details), I obtain

$$P = 0.9316$$

which implies

$$Q = 0.1305, \qquad R = \begin{pmatrix} 0.5444 \\ -0.1729 \\ 0.2908 \\ -0.7092 \\ -0.0391 \\ -0.5328 \end{pmatrix}, \qquad S = \begin{pmatrix} 0.3997 \\ 0.4842 \\ 3.2626 \\ 3.2626 \\ 0.0644 \\ -0.2692 \end{pmatrix}$$

Question 5. See the attached plot.

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12 September 2004
Revised 6 October 2004