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Digression on log-linearization

A simple method for approximating the solution to a dynamic stochastic model is to: (i) computing the non-stochastic steady state, (ii) log-linearize the model around the steady state, and (iii) solving the resulting system of difference equations. To illustrate this procedure, suppose we have a closed economy RBC model. In what follows, I assume that you are comfortable enough with the event-tree notation that you could make this exposition more rigorous if you were so inclined.

Consider the social planning problem of maximizing utility

$$\mathsf{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t [U(c_t) + V(\ell_t)] \right\}$$

subject to a resource constraint

$$c_t + k_{t+1} = z_t F(k_t, n_t) + (1-\delta) k_t$$

with time constraint

$$n_t + \ell_t = 1$$

I assume that period utility is separable between consumption and leisure for expository convenience. Finally, let log technology follows an AR(1),

$$\log(z_{t+1}) = \phi \log(z_t) + \varepsilon_{t+1}, \qquad 0 < \phi < 1$$

where ε_{t+1} is Gaussian white noise. Since this process has unconditional mean of zero, the long run average productivity level is $z = 1$. (I will use unadorned letters to denote steady state values).

The first order conditions associated with this problem include the consumption Euler equation

$$1 = \mathsf{E}_t \left\{ \beta \frac{U'(c_{t+1})}{U'(c_t)} [1 + z_{t+1} F_k(k_{t+1}, n_{t+1}) - \delta] \right\}$$

and the labor supply condition

$$\frac{V'(1-n_t)}{U'(c_t)} = z_t F_n(k_t, n_t)$$

along with the resource constraint.

Non-stochastic steady state

Suppose that we turn off the shocks. Let $z = 1$ and solve for steady state capital, consumption and employment (k, c, n). These satisfy the Euler equation

$$1 = \beta[1 + F_k(k, n) - \delta]$$

the labor supply condition

$$\frac{V'(1-n)}{U'(c)} = F_n(k, n)$$

and the resource constraint.

$$c + \delta k = F(k, n) = y$$

These constitute three non-linear equations in three unknowns. In principle, we could solve them on a computer.

Log-linearization

Define the log-deviation of a variable x_t from its steady state value as

$$\hat{x}_t = \log\left(\frac{x_t}{x}\right)$$

With this notation, a variable is at steady state when its log-deviation is zero. The popularity of this method comes from the units-free nature of the variables. Log-deviations are approximate percentage deviations from steady state and the coefficients of log-linear models are elasticities. Suppose that we have a production function

$$y_t = F(k_t, n_t)$$

Then the log-linear approximation around $\left(\hat{k}_t, \hat{n}_t\right) = 0$ is

$$y\hat{y}_t = F_k(k, n)k\hat{k}_t + F_n(k, n)n\hat{n}_t$$

which is often written

$$\hat{y}_t = \frac{F_k(k, n)k}{F(k, n)}\hat{k}_t + \frac{F_n(k, n)n}{F(k, n)}\hat{n}_t$$

so that the coefficients on the log-deviations $\hat{k}_t$, $\hat{n}_t$ are elasticities. A 1% increase in $\hat{k}_t$ near the steady-state gives approximately a $\frac{F_k(k,n)k}{F(k,n)}100\%$ increase in $\hat{y}_t$.

Let the production function be $y = F(k,n) = k^\theta n^{1-\theta}$ so that

$$\hat{y}_t = \theta \hat{k}_t + (1-\theta)\hat{n}_t$$

Now let the utility functions be

$$\begin{aligned} U(c) &= \frac{c^{1-\sigma}}{1-\sigma} \Longrightarrow \widehat{U'(c_t)} = \frac{U''(c)c}{U'(c)}\hat{c}_t = -\sigma \hat{c}_t \\ V(\ell) &= \frac{\ell^{1-\eta}}{1-\eta} \Longrightarrow \widehat{V'(\ell_t)} = \frac{V''(\ell)\ell}{V'(\ell)}\hat{\ell}_t = -\eta \hat{\ell}_t \end{aligned}$$

Using the time constraint, we also know that $\ell = 1-n$ and that

$$(1-n)\hat{\ell}_t + n\hat{n}_t = 0$$

so that we can also write

$$V'\widehat{(1-n_t)} = \frac{n}{1-n}\eta \hat{n}_t$$

This means that the log-linear versions of our equations are

$$\begin{aligned} 0 &= \mathsf{E}_t \left\{ -\sigma(\hat{c}_{t+1} - \hat{c}_t) + \frac{1}{\beta}[\hat{z}_{t+1} + (\theta-1)(\hat{k}_{t+1} - \hat{n}_{t+1})] \right\} \\ 0 &= \left(\frac{n}{1-n}\eta + \theta \right) \hat{n}_t + \sigma \hat{c}_t - \hat{z}_t - \theta \hat{k}_t \\ 0 &= c\hat{c}_t + k\hat{k}_{t+1} - y\hat{z}_t - [\theta y + (1-\delta)k]\hat{k}_t - (1-\theta)y\hat{n}_t \\ \hat{z}_{t+1} &= \phi \hat{z}_t + \varepsilon_{t+1} \end{aligned}$$

Method of undetermined coefficients

We now have a system of linear difference equations. Let me introduce the notation

$$X_t = \hat{k}_{t+1}, \qquad Y_t = \begin{pmatrix} c_t \\ n_t \end{pmatrix}, \qquad Z_t = \hat{z}_t$$

(Notice the timing convention! This is because $\hat{k}_{t+1}$ is chosen at date t). Following the notation in Uhlig (1999), we have the system of equations

$$\begin{aligned} 0 &= AX_t + BX_{t-1} + CY_t + DZ_t \\ 0 &= \mathsf{E}_t\{FX_{t+1} + GX_t + HX_{t-1} + JY_{t+1} + KY_t + LZ_{t+1} + MZ_t\} \\ Z_{t+1} &= NZ_t + \varepsilon_{t+1} \end{aligned}$$

where the first equation captures the "static" resource constraint and labor supply conditions so that A and B are 2-by-1, C is 2-by-2, D is 2-by-1. The second equation captures the "forward-looking" consumption Euler equation, and the last is the law of motion for the exogenous shocks.

This notation is a little too general. For this particular model, the coefficients G, H and M are all zero so that in what follows I will simply write the forward looking equation as

$$0 = \mathsf{E}_t\{FX_{t+1} + JY_{t+1} + KY_t + LZ_{t+1}\}$$

The key step in this procedure is to "guess" that solutions to the model take the form

$$\begin{aligned} X_t &= PX_{t-1} + QZ_t \\ Y_t &= RX_{t-1} + SZ_t \end{aligned}$$

for unknown coefficient matrices P, Q, R, S. For this model, P and Q are scalars while R and S are 2-by-1. We now establish what restrictions the RBC model puts on these unknown coefficient matrices. To do this, plug the guesses into the system of static equations to get

$$\begin{aligned} 0 &= A(PX_{t-1} + QZ_t) + BX_{t-1} + C(RX_{t-1} + SZ_t) + DZ_t \\ &= (AP + B + CR)X_{t-1} + (AQ + D + CS)Z_t \end{aligned}$$

But for this to hold identically for any state X_{t-1} and any shock Z_t, it must be the case that

$$\begin{aligned} 0 &= AP + B + CR \\ 0 &= AQ + D + CS \end{aligned}$$

Supposing that the 2-by-2 matrix C is invertible, this is the same as

$$\begin{aligned} R &= -C^{-1}(AP+B) \\ S &= -C^{-1}(AQ+D) \end{aligned}$$

So once we have solved for the coefficients for the evolution of the endogenous state variables, namely P and Q, we can easily back out the coefficients for the evolution of the control variables. Now plugging our guesses into the system of Euler equations gives

$$\begin{aligned} 0 &= \mathsf{E}_t\{FX_{t+1} + JY_{t+1} + KY_t + LZ_{t+1}\} \\ &= \mathsf{E}_t\{F(PX_t + QZ_{t+1}) + J(RX_t + SZ_{t+1}) + K\,(RX_{t-1} + SZ_t) + LZ_{t+1}\} \\ &= \mathsf{E}_t\{F(P(PX_{t-1} + QZ_t) + QZ_{t+1}) + J(R(PX_{t-1} + QZ_t) + SZ_{t+1}) + K\,(RX_{t-1} + SZ_t) + LZ_{t+1}\} \\ &= \mathsf{E}_t\{FP^2X_{t-1} + FPQZ_t + FQZ_{t+1} + JRPX_{t-1} + JRQZ_t + JSZ_{t+1} + KRX_{t-1} + KSZ_t + LZ_{t+1}\} \end{aligned}$$

Now taking the conditional expectations

$$0 = FP^2X_{t-1} + FPQZ_t + FQNZ_t + JRPX_{t-1} + JRQZ_t + JSNZ_t + KRX_{t-1} + KSZ_t + LNZ_t$$

And on collecting terms

$$0 = (FP^2 + JRP + KR)X_{t-1} + (FPQ + FQN + JRQ + JSN + KS + LN)Z_t$$

Hence our restrictions are

$$\begin{aligned} 0 &= FP^2 + JRP + KR \\ 0 &= FPQ + FQN + JRQ + JSN + KS + LN \end{aligned}$$

Now plug the equation $R = -C^{-1}(AP+B)$ into the first set of restrictions to get

$$\begin{aligned} 0 &= FP^2 - J(C^{-1}(AP+B))P - K(C^{-1}(AP+B)) \\ &= \left(F - JC^{-1}A\right)P^2 - \left(JC^{-1}B + KC^{-1}A\right)P - KC^{-1}B \end{aligned}$$

This is a quadratic equation in the unknown P. We solve it and choose the **stable root** (this is equivalent to imposing a **transversality condition**). Once we have determined P, we also have $R = -C^{-1}(AP + B)$.

Turning now to the second set of restrictions from the forward-looking equation, we use the fact that Q is a scalar and that $S = -C^{-1}(AQ + D)$ to write

$$\begin{aligned} 0 &= (FP + FN + JR)Q - J(C^{-1}(AQ + D))N - KC^{-1}(AQ + D) + LN \\ &= (FP + FN + JR - JC^{-1}AN - KC^{-1}A)Q - JC^{-1}DN - KC^{-1}D + LN \end{aligned}$$

or

$$Q = \frac{JC^{-1}DN + KC^{-1}D - LN}{FP + FN + JR - JC^{-1}AN - KC^{-1}A}$$

which we can compute because we have already solved for P and R and all the other coefficients were known to begin with. Finally, we can recover the remaining coefficient via $S = -C^{-1}(AQ+D)$.

So, after all that, we have a method for solving for the unknown coefficients. To recap: we solve for the non-stochastic steady state and construct the known matrices of coefficients. We then solve a quadratic equation for the critical P. We then back out the associated R so that we have all the coefficients on the endogenous state variables. Using P and R we solve for the coefficients on the shocks, Q and S. And then we're done and ready to do interesting things like simulating the model by iterating on the linear laws of motion

$$\begin{aligned} X_t &= PX_{t-1} + QZ_t \\ Y_t &= RX_{t-1} + SZ_t \\ Z_{t+1} &= NZ_t + \varepsilon_{t+1} \end{aligned}$$

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16 August 2004