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International real business cycle models

This note outlines a multi-country version of the standard RBC model. Let there be countable dates, $t = 0, 1, 2...$ and let a state of nature be indexed by ϵ_t. A history is a vector $\epsilon^t = (\epsilon_0, \epsilon_1, ..., \epsilon_t) = (\epsilon^{t-1}, \epsilon_t)$. The unconditional probability of a history ϵ^t being realized as of date zero is denoted $\pi_t(\epsilon^t)$. The initial state ϵ_0 is known as of date zero. We will discuss the nature of the shocks below. At each date and state there is a single consumption good.

- *Preferences*: Let there be I countries with $i \in \{1, 2, ..., I\}$. The representative consumer in country i has preferences over streams of the consumption good $c^i = \{c_t^i(\epsilon^t)\}_{t=0}^{\infty}$ and non-market time ("leisure"), denoted $\ell^i = \{\ell_t^i(\epsilon^t)\}_{t=0}^{\infty}$. These preferences are given by the expected utility function

$$u(c^i, \ell^i) = \sum_{t=0}^{\infty} \sum_{\epsilon^t} \beta^t U[c_t^i(\epsilon^t), \ell_t^i(\epsilon^t)]\pi_t(\epsilon^t), \qquad 0 < \beta < 1$$

  We will typically assume the isoelastic period utility function

$$U(c, \ell) = \frac{(c^{\mu}\ell^{1-\mu})^{1-\gamma}}{1-\gamma}, \qquad 0 < \mu < 1,\ \gamma > 0$$

  We will normalize the representative consumer's endowment of hours available for work to 1 so that in each country, leisure plus labor satisfies

$$\ell_t^i(\epsilon^t) + n_t^i(\epsilon^t) \leq 1$$

  for each date and state. We can alternatively write the representative consumer's preferences in terms of consumption and employment, namely

$$U(c, 1-n) = \frac{[c^{\mu}(1-n)^{1-\mu}]^{1-\gamma}}{1-\gamma}$$

  Notice that with this formulation, labor is a "bad" (the marginal utility of labor supply is negative).

- *Technology*: The single good can be produced in each country according to an aggregate constant returns to scale production function that uses capital and labor

$$y_t^i(\epsilon^t) = z_t^i(\epsilon^t)F[k_t^i(\epsilon^{t-1}), n_t^i(\epsilon^t)]$$

Notice that capital available at the beginning of period t is pre-determined, it only depends on the history ϵ^{t-1}. We will typically assume the Cobb-Douglas functional form

$$F(k,n) = k^\theta n^{1-\theta}, \qquad 0 < \theta < 1$$

The single good can be consumed, bought by the government, or invested. The world resource constraint is

$$\sum_i [c_t^i(\epsilon^t) + x_t^i(\epsilon^t) + g_t^i(\epsilon^t)] \leq \sum_i z_t^i(\epsilon^t)F[k_t^i(\epsilon^{t-1}), n_t^i(\epsilon^t)]$$

where $x_t^i(\epsilon^t)$ denotes total investment.

- *Net exports*: We do not require that consumption plus investment plus government spending equal output for each country. If we did so, we would be modelling I **closed** economies. Instead, for each country net exports are

$$y_t^i(\epsilon^t) - [c_t^i(\epsilon^t) + x_t^i(\epsilon^t) + g_t^i(\epsilon^t)]$$

So an alternative statement of the world resource constraint is that the sum of net exports across countries must be zero.

- *Time-to-build*: Additions to the capital stock require investment expenditures for J time periods. So

$$k_{t+1}^i(\epsilon^t) = (1-\delta)k_t^i(\epsilon^{t-1}) + s_t^i(1, \epsilon^t), \qquad 0 < \delta < 1$$

and

$$s_{t+1}^i(j, \epsilon^t) = s_t^i(j+1, \epsilon^t), \qquad j = 1, ..., J-1$$

where δ denotes the physical depreciation rate of capital and $s_t^i(j, \epsilon^t)$ denotes the number of investment projects in country i that will mature in j periods time. Suppose that over the length of the project, a fraction $\phi(j)$ (with $\sum_j \phi(j) = 1$) of total investment must be spent on a project of type j in order to ensure completion. Total investment is then

$$x_t^i(\epsilon^t) = \sum_{j=1}^{J-1} \phi(j) s_t^i(j, \epsilon^t)$$

Different assumptions about $\phi(j)$ reflect different assumptions about the nature of project completion. For example, if $J = 2$, then one assumption might be

$$\phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which would mean that all the investment is done up front and the project will add to the economy's capital stock in $J = 2$ periods time. Alternatively, if

$$\phi = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Then half the investment is up-front and another half has to be made in the next period before the project is finalized. Typically, we will assume the uniform distribution $\phi(j) = 1/J$ each j. The conventional law of motion (without time-to-build) is the special case $J = 1$ so that capital accumulation would be

$$k_{t+1}^i(\epsilon^t) = (1 - \delta) k_t^i(\epsilon^{t-1}) + x_t^i(\epsilon^t)$$

- *Shocks*: There are two sources of uncertainty, technology shocks and government expenditure shocks. Let $z_t = \{z_t^i(\epsilon^t)\}_{i=1}^I$ and $g_t = \{g_t^i(\epsilon^t)\}_{i=1}^I$ denote vectors of realizations of the shocks. These will evolve according to vector autoregressions of the form

$$\begin{aligned} \log(z_{t+1}) &= A \log(z_t) + \varepsilon_{t+1}^z \\ \log(g_{t+1}) &= B \log(g_t) + \varepsilon_{t+1}^g \end{aligned}$$

where A and B are fixed matrices of coefficients and the innovations ε_{t+1}^z and ε_{t+1}^g are independent and normally distributed with constant variance/covariance matrices V^z and V^g. With this notation, $\epsilon_t = (\varepsilon_t^z, \varepsilon_t^g)$.

Social planner's problem

We study the social planner's problem. Suppose that the social planner attaches equal welfare weights $1/I$ to each country i. Then the planner's problem is to choose allocations to maximize

$$\frac{1}{I}\sum_i \sum_{t=0}^{\infty} \sum_{\epsilon^t} \beta^t U[c_t^i(\epsilon^t), \ell_t^i(\epsilon^t)]\pi_t(\epsilon^t)$$

subject to resource constraints and the constraint on the time endowments. For expositional simplicity, suppose that $J = 1$. Then

$$x_t^i(\epsilon^t) = s_t^i(1, \epsilon^t)$$

and the world resource constraint can be written

$$\sum_i \{c_t^i(\epsilon^t) + k_{t+1}^i(\epsilon^t) + g_t^i(\epsilon^t)\} \leq \sum_i \{z_t^i(\epsilon^t)F[k_t^i(\epsilon^{t-1}), n_t^i(\epsilon^t)] + (1-\delta)k_t^i(\epsilon^{t-1})\}$$

Let the Lagrange multiplier associated with the date t and history ϵ^t resource constraint be $\lambda_t(\epsilon^t)$. Then the Lagrangian for the social planner is

$$\begin{aligned}\mathcal{L} &= \sum_i \sum_{t=0}^{\infty} \sum_{\epsilon^t} \beta^t U[c_t^i(\epsilon^t), 1 - n_t^i(\epsilon^t)]\pi_t(\epsilon^t) \\ &\quad + \sum_{t=0}^{\infty} \sum_{\epsilon^t} \lambda_t(\epsilon^t) \left[\sum_i \{z_t^i(\epsilon^t)F[k_t^i(\epsilon^{t-1}), n_t^i(\epsilon^t)] + (1-\delta)k_t^i(\epsilon^{t-1})\} - \sum_i \{c_t^i(\epsilon^t) + k_{t+1}^i(\epsilon^t) + g_t^i(\epsilon^t)\} \right]\end{aligned}$$

The first order conditions for this problem include, for each i, t and ϵ^t,

$$\frac{\partial \mathcal{L}}{\partial c_t^i(\epsilon^t)} = 0 \Longleftrightarrow \beta^t U_{c,t}^i(\epsilon^t)\pi_t(\epsilon^t) = \lambda_t(\epsilon^t)$$

$$\frac{\partial \mathcal{L}}{\partial n_t^i(\epsilon^t)} = 0 \Longleftrightarrow \beta^t U_{\ell,t}^i(\epsilon^t)\pi_t(\epsilon^t) = \lambda_t(\epsilon^t) z_t^i(\epsilon^t) F_{n,t}^i(\epsilon^t)$$

and

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}^i(\epsilon^t)} = 0 \Longleftrightarrow \lambda_t(\epsilon^t) = \sum_{\epsilon'} \lambda_{t+1}(\epsilon^t, \epsilon')\{z_{t+1}^i(\epsilon^t, \epsilon')F_{k,t+1}^i(\epsilon^t, \epsilon') + 1 - \delta\}$$

where the notation $U_{c,t}^i(\epsilon^t)$, for example, is shorthand for $U_c[c_t^i(\epsilon^t), 1 - n_t^i(\epsilon^t)]$. The usual complete market risk-sharing outcome makes its presence felt in the first condition, namely

$$U_{c,t}^i(\epsilon^t) = \frac{\lambda_t(\epsilon^t)}{\beta^t \pi_t(\epsilon^t)}$$

and hence marginal utilities are equalized across countries (marginal utilities are the same since each have same welfare weight; if the planner gave different weights to different countries, the marginal utilities would be equalized up to a constant of proportionality). Notice that in general this does not mean that consumptions are equalized across countries. Marginal utility depends on both consumption and leisure different combinations of consumption and leisure can give rise to the same marginal utility.

The first order conditions can be re-written

$$\frac{U^i_{\ell,t}(\epsilon^t)}{U^i_{c,t}(\epsilon^t)} = z^i_t(\epsilon^t)F^i_{n,t}(\epsilon^t)$$

and

$$U^i_{c,t}(\epsilon^t) = \sum_{\epsilon'} \beta U^i_{c,t+1}(\epsilon^t,\epsilon')\frac{\pi_{t+1}(\epsilon^t,\epsilon')}{\pi_t(\epsilon^t)}\{z^i_{t+1}(\epsilon^t,\epsilon')F^i_{k,t+1}(\epsilon^t,\epsilon') + 1 - \delta\}$$

The first condition requires the equality of the marginal rate of substitution between leisure and consumption with the marginal product of labor. The second condition is the consumption Euler equation with the gross return on capital maturing in the next period.

From now on, suppress the ϵ^t notation, so these are just

$$\frac{U^i_{\ell,t}}{U^i_{c,t}} = z^i_t F^i_{n,t}$$

and

$$U^i_{c,t} = \mathsf{E}_t\left\{\beta U^i_{c,t+1}[1 + z^i_t F^i_{k,t+1} - \delta]\right\}$$

We can solve a model like this by (i) computing the non-stochastic steady state, (ii) log-linearizing the model around the steady state and (iii) solving the resulting system of difference equations. I discuss this method in more detail in a companion note.

Notes on quantity puzzles

Backus et al (1995) study quarterly data from 1970 to 1990 on 10 industrialized countries. With respect to quantities, their main focus is on relative volatilities of consumption, employment, investment, productivity and the trade balance as well as on the cross-country correlations of these variables. Essentially all of their calculations refer to data detrended with the **Hodrick-Prescott filter**. Their main findings are:

- Standard deviations of output fluctuations on the order of 1.0-2.0%. Standard deviations of net export fluctuations on the order of 0.5-1.5%.

- For each country, consumption is the same or less volatile than output.
- Except for Austria, employment is the same or less volatile than output.
- But investment is 2-3 times more volatile than output.
- Total factor productivity (measured by Solow residuals) is less volatile than output.
- Output fluctuations are persistent, with first order serial correlation coefficients of around 0.60-0.90 in quarterly data.
- Consumption, investment, productivity, and employment all tend to be procyclical, but net exports tends to be **countercyclical**.
- Output movements of all countries are correlated with the United States, with contemporaneous correlation coefficients of around 0.60.
- But **consumption is less correlated** across countries than output is, with coefficients around 0.50.
- Similarly, productivity is less correlated across countries than output is, with coefficients around 0.55.

These last two findings are strongly at odds with their 2-country IRBC model. Backus et al's benchmark model has capital accumulation with the time-to-build feature, but only technology shocks (i.e., no government expenditure shocks). Productivity shocks are mildly correlated across countries. They compare their model to data on the US and an aggregate of European countries. They find that the model does all right at matching the empirical volatilities of output and the relative volatilities of employment and productivity, but it leads to net exports and investment that are much too volatile. Moreover the model predicts

- Cross-country output, investment and employment correlations that are negative (they are positive in the data).
- Cross-country consumption correlations that are too high.

Backus et al consider other variants of their model. For example, if they add transportation costs they can reduce the excess volatility of net exports and investment, but this modification has little effect on cross country correlations of output or consumption. Also, it makes net exports even more procyclical than before.

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