316-632 INTERNATIONAL MONETARY ECONOMICS NOTE 1a

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Basic concepts and the small open economy

This note uses a model of a small open economy perfectly integrated into world capital markets to introduce some basic concepts of international macroeconomics. Neither this nor any of my other notes is intended as a substitute for reading the textbook. We will consider a very simple environment: a discrete time model of an economy populated by a single representative consumer with an infinite horizon. For the moment, we will assume that there is no uncertainty. The primary purpose of this model is to develop an understanding of the way that international borrowing and lending can be used to **smooth consumption**.

- *Preferences*: The representative consumer has a time-additive utility function over an infinite stream of consumption, $c = \{c_t\}_{t=0}^{\infty}$. Specifically

$$u(c) = \sum_{t=0}^{\infty} \beta^t U(c_t)$$

  with constant time discount factor $0 < \beta < 1$. The period utility function $U(c)$ is assumed to be strictly increasing and concave.

- *Endowments*: There is no production. Instead, there is simply an exogenously given supply of the consumption good at each date, $y = \{y_t\}_{t=0}^{\infty}$ and a given stock of initial assets, B_0.
- *Flow constraints*: There is an exogenously given world real interest rate $r > 0$ and the country can freely borrow or lend at this rate. If the country earns more than it consumes, $y_t - c_t > 0$, it runs a **trade surplus**. This is used to acquire claims on foreign assets (say bonds) that mature in one period's time, $B_{t+1} > 0$, that yield gross interest $1+r$. Similarly, if the country runs a trade deficit, $y_t - c_t < 0$, it covers this by selling claims on its future earnings to the rest of the world, $B_{t+1} < 0$. Denoting the beginning-of-period net claims on foreign assets by B_t, the **flow budget constraint** of the country is for each t

$$c_t + B_{t+1} = (1 + r)B_t + y_t, \qquad B_0 \text{ given} \tag{1}$$

  The **current account** balance is

$$B_{t+1} - B_t = rB_t + y_t - c_t$$

i.e., the trade balance plus or minus net interest earnings.

- *No-Ponzi-games*: Impose a constraint that rules out certain explosive borrowing schemes

$$\lim_{t\to\infty}\left(\frac{1}{1+r}\right)^t B_{t+1} \geq 0$$

- *Intertemporal constraint*: The flow budget constraints and the no-Ponzi game condition together imply an **intertemporal budget constraint**. This is calculated by iterating, as follows

$$\begin{aligned} B_0 &= \frac{1}{1+r}(c_0 - y_0) + \frac{1}{1+r}B_1 \\ B_1 &= \frac{1}{1+r}(c_1 - y_1) + \frac{1}{1+r}B_2 \end{aligned}$$

Putting these together

$$B_0 = \frac{1}{1+r}(c_0 - y_0) + \left(\frac{1}{1+r}\right)^2 (c_1 - y_1) + \left(\frac{1}{1+r}\right)^2 B_2$$

But notice

$$B_2 = \frac{1}{1+r}(c_2 - y_2) + \frac{1}{1+r}B_3$$

so once again

$$B_0 = \frac{1}{1+r}(c_0 - y_0) + \left(\frac{1}{1+r}\right)^2 (c_1 - y_1) + \left(\frac{1}{1+r}\right)^3 (c_2 - y_2) + \left(\frac{1}{1+r}\right)^3 B_3$$

Repeating these arguments T times gives

$$(1+r)B_0 = \sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t (c_t - y_t) + \left(\frac{1}{1+r}\right)^T B_{T+1}$$

And now taking the horizon $T \to \infty$ and imposing the no-Ponzi-game condition gives the intertemporal budget constraint

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t c_t \leq (1+r)B_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t$$

This says that the present value of consumption must be less than the present value of wealth. It is quite common to simplify these formulas by defining the **present value price**, i.e., the price as of date 0 of a unit of consumption in date t by

$$p_t = \left(\frac{1}{1+r}\right)^t$$

Notice that $0 < p_t \leq 1$. A unit of consumption today is worth a unit of consumption today, $p_0 = 1$. But a unit of consumption tomorrow is worth less than a unit of consumption today because I can lend 1 unit and get $1+r$ back tomorrow. Hence a unit of consumption tomorrow is worth only $p_1 = \frac{1}{1+r} < 1$. And so on: a unit of consumption far into the future is worth almost nothing. With this shorthand, the intertemporal budget constraint can also be written more compactly

$$\sum_{t=0}^{\infty} p_t c_t \leq (1+r)B_0 + \sum_{t=0}^{\infty} p_t y_t \tag{2}$$

- *Optimization*: There are two equivalent procedures:

Intertemporal budget constraint. Use the intertemporal budget constraint given in equation (2). There is a **single** Lagrange multiplier $\lambda \geq 0$ to go along with the single budget constraint and the Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t) + \lambda \left[(1+r)B_0 + \sum_{t=0}^{\infty} p_t(y_t - c_t)\right]$$

The key first order condition for consumption c_t at each date t is

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Longleftrightarrow \beta^t U'(c_t) = \lambda p_t$$

The marginal utility of consumption is set equal the price of consumption. Now consider consumption choices at two adjacent dates

$$\begin{aligned} \beta^t U'(c_t) &= \lambda p_t \\ \beta^{t+1} U'(c_{t+1}) &= \lambda p_{t+1} \end{aligned}$$

Putting these together

$$\frac{\beta U'(c_{t+1})}{U'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{1+r}$$

The marginal rate of substitution of consumption between dates is equal to the relative price.

Sequence of flow constraints. Use the sequence of flow budget constraints as given in equation (1). There is a **sequence** of Lagrange multipliers $\{\eta_t\}_{t=0}^{\infty}$, one for each flow budget constraint, and the Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t) + \sum_{t=0}^{\infty} \eta_t \left[(1+r)B_t + y_t - c_t - B_{t+1}\right]$$

We now need first order conditions for consumption c_t and bond acquisitions B_{t+1} at each date t. These are given by

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Longleftrightarrow \beta^t U'(c_t) = \eta_t$$

and

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Longleftrightarrow \eta_t = (1+r)\eta_{t+1}$$

Putting these together

$$\beta^t U'(c_t) = \eta_t = (1+r)\eta_{t+1} = (1+r)\beta^{t+1} U'(c_{t+1})$$

and on rearranging

$$\frac{\beta U'(c_{t+1})}{U'(c_t)} = \frac{\eta_{t+1}}{\eta_t} = \frac{1}{1+r}$$

Again, the marginal rate of substitution of consumption between dates is set equal to the relative price. Notice that the two approaches are equivalent with the Lagrange multipliers in the second approach related to the single multiplier and present value price in the first approach by

$$\eta_t = \lambda p_t = \lambda \left(\frac{1}{1+r}\right)^t$$

The first order conditions of this and similar problems are often called **consumption Euler equations**.

- *Consumption smoothing*: What does the path of consumption look like? Write the consumption Euler equation as

$$U'(c_t) = \beta(1+r)U'(c_{t+1})$$

A consumer that gives up one unit of consumption gives up $U'(c_t)$ utility, but that unit of foregone consumption leads to $1+r$ units of consumption tomorrow which is worth $\beta U'(c_{t+1})$ in utility. At an optimum, the marginal costs and benefits of foregone consumption should be equal. For future reference, write $0 < \beta < 1$ as $\beta \equiv \frac{1}{1+\rho}$ where $\rho > 0$ is known as the **time discount rate**. We have three possibilities to deal with. If $\beta(1+r) = 1$, the time discount rate $\rho = r$ the world real interest rate and optimal consumption requires $U'(c_t) = U'(c_{t+1})$ for any two dates. But $U'(c_t) > 0$ so this means $c_t = c_{t+1}$ all t. In short, if $\beta(1+r) = 1$, then consumption is flat. Similarly, if $\beta(1+r) > 1$, the representative consumer is patient (relative to the interest rate) and so $U'(c_t) > U'(c_{t+1})$. But since $U''(c_t) < 0$, this implies $c_t < c_{t+1}$ so consumption is growing over time: a relatively patient consumer is willing to save today and consume more tomorrow and so has growing consumption. The converse is true if $\beta(1+r) < 1$, an impatient consumer is not willing to save and has falling consumption. To summarize

$$
\begin{aligned}
\beta(1+r) &= 1 \Longrightarrow U'(c_t) = U'(c_{t+1}) \Longrightarrow c_t = c_{t+1} \\
\beta(1+r) &> 1 \Longrightarrow U'(c_t) > U'(c_{t+1}) \Longrightarrow c_t < c_{t+1} \\
\beta(1+r) &< 1 \Longrightarrow U'(c_t) < U'(c_{t+1}) \Longrightarrow c_t > c_{t+1}
\end{aligned}
$$

A parametric example might help. Let the period utility function have the **constant intertemporal elasticity of substitution** (CIES) form

$$U(c) \equiv \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \qquad \sigma > 0$$

with elasticity σ. Then marginal utility is $U'(c_t) = c_t^{-\frac{1}{\sigma}}$ and the consumption Euler equation can be written

$$\left(\frac{c_{t+1}}{c_t}\right) = [\beta(1+r)]^{\sigma}$$

Hence consumption is growing, constant, or falling depending on whether $\beta(1+r) \gtreqless 1$.

- *Solving for the level of consumption*: While the consumption Euler equation helps us characterize the **growth rate** of consumption, to solve for its **level** we need to use the intertemporal budget constraint. Here are two examples of complete solutions.

1. Let $\beta(1+r) = 1$, so that the consumer's time discount rate is equal to the world real interest rate. Then the results above imply that consumption is a constant $c_t = c_{t+1} = \bar{c}$ all t. But if

consumption is a constant, the intertemporal budget constraint implies

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \bar{c} = (1+r)B_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t$$

The geometric sum on the left hand side reduces to

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \bar{c} = \frac{1+r}{r}\bar{c}$$

Hence

$$\bar{c} = rB_0 + \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t \tag{3}$$

This is a statement of the **permanent income hypothesis**: the level of consumption is equal to a fraction $\frac{r}{1+r}$ of the entire present value of wealth ("permanent income"). An important special case is when $y_t = y_0$ all t. Then $\bar{c} = rB_0 + y_0$. Consumption is equal to the constant flow of y_0 plus or minus the annuity value of initial assets rB_0.

2. Let $U'(c_t) = c_t^{-\frac{1}{\sigma}}$. Then the consumption Euler equation is, once again,

$$\left(\frac{c_{t+1}}{c_t}\right) = [\beta(1+r)]^{\sigma}$$

Hence

$$c_t = [\beta(1+r)]^{\sigma t}\, c_0$$

We now need to pin down the initial consumption level, c_0. The intertemporal budget constraint gives

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t [\beta(1+r)]^{\sigma t}\, c_0 = (1+r)B_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t$$

or equivalently

$$c_0\sum_{t=0}^{\infty}\left[\beta^{\sigma}(1+r)^{\sigma-1}\right]^t = (1+r)B_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t$$

and on simplifying

$$c_0 = (v+r)B_0 + \frac{v+r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t$$

where

$$v \equiv 1 - \beta^{\sigma}(1+r)^{\sigma}$$

Notice that if $\beta(1+r) = 1$, the composite parameter is $v = 0$ and we have the previous solution $c_0 = \bar{c}$ as given above in equation (3). If $\beta(1+r) > 1$, then $v < 0$ and $c_0 < \bar{c}$ while if $\beta(1+r) < 1$, $v > 0$ and $c_0 > \bar{c}$.

The current account

Define the **permanent** level of output $\bar{y}$ as the number that solves

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \bar{y} = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t$$

Then

$$\bar{y} = \frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t$$

The permanent value $\bar{y}$ denotes an appropriately weighted average of the sequence $\{y_t\}_{t=0}^{\infty}$. Recall that the current account balance is given by

$$B_{t+1} - B_t = rB_t + y_t - c_t$$

So if, as in equation (3), $\bar{c} = rB_0 + \bar{y}$, we have at date $t = 0$

$$B_1 - B_0 = y_0 - \bar{y}$$

The economy runs a current account balance if it has a high value of y_0 relative to the permanent value $\bar{y}$. Alternatively, when $y_0 < \bar{y}$, the economy runs a current account deficit.

More generally, once we have solved for consumption c_t, we can use the flow constraints and the initial condition B_0 to solve for the stock of bonds at any date t

$$B_t = (1+r)^t B_0 + \sum_{k=0}^{t} (1+r)^{t-k}(y_k - c_k)$$

The net asset position at time t is the compounded return on the initial assets plus the compounded value of trade balances.

Time-varying world interest rates

If the economy faces a time-varying sequence $\{r_t\}_{t=0}^{\infty}$ of given world interest rates instead of the single constant r, the analysis proceeds in the same manner as before except that the present value prices used to form the intertemporal budget constraint are now given by

$$p_t = \prod_{s=0}^{t} \frac{1}{1+r_s}$$

The consumption Euler equations are now of the form

$$\frac{\beta U'(c_{t+1})}{U'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{1+r_{t+1}}$$

But, otherwise not much changes.

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