316-406 ADVANCED MACRO TECHNIQUES **Midterm Solutions**

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This exam lasts **90 minutes** and has three questions, each of equal marks. Within each question there are a number of parts and the weight given to each part is also indicated. You have **10 minutes perusal** before you can start writing answers.

Question 1. *Stochastic Solow Growth Model* (30 marks): Let time be discrete, $t = 0, 1, ...$ Let the national resource constraint be

$$c_t + i_t = y_t = z_t f(k_t)$$

where c_t denotes consumption, i_t denotes investment, y_t denotes output, and k_t denotes capital. The production function has the properties

$$\begin{aligned} f(0) &= 0 \\ f'(k) &> 0, \qquad f''(k) < 0 \\ \lim_{k\to 0} f'(k) &= \infty, \qquad \lim_{k\to\infty} f'(k) = 0 \end{aligned}$$

The production function is buffeted by random IID technology shocks $\{z_t\}$ with properties to be described below and with a given initial condition z_0. Let capital accumulation be given by

$$k_{t+1} = i_t, \qquad k_0 \text{ given}$$

(i.e., there is "full depreciation"). Finally, let consumption be a fixed fraction of national output

$$c_t = (1 - s)y_t, \qquad 0 < s < 1$$

where s denotes the national saving rate.

(a) (6 marks): Show that this model can be reduced to a single non-linear stochastic difference equation in k_t and z_t, i.e., that you can write the model as

$$k_{t+1} = \psi(k_t, z_t), \qquad k_0 \text{ and } z_0 \text{ given}$$

Provide an explicit formula for the function ψ.

Solution: Straightforward manipulations give

$$\begin{aligned} k_{t+1} = i_t &= z_t f(k_t) - c_t \\ &= z_t f(k_t) - (1-s) z_t f(k_t) \\ &= s z_t f(k_t) \end{aligned}$$

So we have a single non-linear difference equation

$$k_{t+1} = s z_t f(k_t) \equiv \psi(k_t, z_t)$$

with given initial conditions k_0 and z_0.

(b) (6 marks): Let $z_t = \bar{z} = 1$ always and let $\bar{k}$ denote a solution to

$$\bar{k} = \psi(\bar{k}, 1)$$

How many such $\bar{k}$ are there? Linearize the function $\psi(k_t, 1)$ around each of these points and determine the local stability or instability of each such point.

Solution: Let $\bar{k}$ denote a solution to

$$\bar{k} = \psi(\bar{k}, 1) = s f(\bar{k})$$

There a two such points one is $\bar{k} = 0$, since $0 = sf(0) = 0$ and the other is interior, $\bar{k} > 0$. The interior point is found as the intersection of the 45-degree line through the origin with the concave function $sf(k)$ (concave since $f(k)$ is concave and s is a positive constant). The linearization is

$$k_{t+1} \simeq \bar{k} + s f'(\bar{k})(k_t - \bar{k})$$

Since $\lim_{\bar{k} \to 0} f'(\bar{k}) = \infty$, the trivial steady state is unstable. Since $sf(k)$ cuts the 45-degree line from above, $0 < sf'(\bar{k}) < 1$, so the interior steady state is (locally) stable.

(c) (6 marks): Suppose that the production function is Cobb-Douglas with capital share α,

$$f(k) \equiv k^{\alpha}, \qquad 0 < \alpha < 1$$

Provide an explicit solution for each $\bar{k}$ (continuing to hold $z_t = \bar{z} = 1$). Explain how the fixed points depend on the parameters α and s. Given economic interpretations.

Solution: Clearly one solution is the trivial $\bar{k} = 0$ which does not depend on any parameters and can be ignored. The other is the solution to

$$\bar{k} = s\bar{k}^{\alpha}$$

or

$$\log(\bar{k}) = \frac{1}{1-\alpha}\log(s)$$

Clearly the steady state capital stock is higher when national savings, s, is higher and is higher the more intensively capital is used in production.

(e) (12 marks): Suppose that $\log(z_t)$ are IID Gaussian with mean 0 and variance σ^2. Let

$$\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right)$$

be the log deviation of some variable from its "non-stochastic steady state". Log-linearize $\psi(k_t, z_t)$ to derive an approximate linear stochastic difference equation in the state $\hat{k}_t$ and the shocks $\hat{z}_t$ (assume as above that $f(k)$ is Cobb-Douglas). Solve for the stationary distribution of $\hat{k}_t$ and explain how its mean and variance depend on the parameters α, s and σ. Give economic interpretations.

Solution: Write the stochastic difference equation as

$$k_{t+1} = s z_t k_t^{\alpha} \equiv \psi(k_t, z_t)$$

There's actually no need to approximately log-linearize this. The model is already exactly log-linear! Take logs to get

$$\log(k_{t+1}) = \log(s) + \log(z_t) + \alpha\log(k_t)$$

Now subtract $\log(\bar{k})$ from both sides

$$\log(k_{t+1}) - \log(\bar{k}) = \log(s) + \log(z_t) + \alpha\log(k_t) - \log(\bar{k})$$

$$
\begin{aligned}
&= \log(s) + \log(z_t) + \alpha[\log(k_t) - \log(\bar{k})] - (1-\alpha)\log(\bar{k}) \\
&= \log(s) + \log(z_t) + \alpha[\log(k_t) - \log(\bar{k})] - (1-\alpha)\frac{1}{1-\alpha}\log(s) \\
&= \log(z_t) + \alpha[\log(k_t) - \log(\bar{k})]
\end{aligned}
$$

or

$$\hat{k}_{t+1} = \alpha \hat{k}_t + \hat{z}_t$$

(which uses $\log(\bar{z}) = 0$ so $\hat{z}_t = \log(z_t)$). If you had log-linearized you would get the same result. Standard calculations then give that the stationary distribution for the log deviation $\hat{k}_t$ is normal with mean

$$\mathsf{E}\{\hat{k}_t\} = \frac{1}{1-\alpha}\mathsf{E}\{\hat{z}_t\} = 0$$

and variance

$$\mathsf{Var}\{\hat{k}_t\} = \frac{1}{1-\alpha^2}\mathsf{Var}\{\hat{z}_t\} = \frac{\sigma^2}{1-\alpha^2}$$

On average, the log-deviations of the capital stock from their steady state value are zero: remember that these are deviations and if the deviations from the steady state are not zero on average, maybe we should focus our attention elsewhere. The variance around this mean is increasing both in the variance σ^2 of the technology shocks but also in the persistence of the capital stock. That is, when $\alpha \to 1$, the production function becomes nearly linear and deviations from steady state are slow to die out. The more persistent these deviations, the higher the long run variance of the distribution of the capital stock.

Question 2. *Stochastic Labor Demand* (30 marks): Suppose that a firm faces a stochastic real wage rate each period which follows an m-state Markov chain (w, P, μ_0) where w is an m-vector, P is a transition matrix and μ_0 is an initial distribution. Suppose that each period, the firm solves the static profit maximization problem over employment

$$\pi(w) = \max_{n} \{f(n) - wn\}$$

where w is this period's random wage realization and n is the firm's labor force. Suppose that $f(n)$ is strictly increasing and strictly concave in n.

(a) (5 marks): Explain how a labor demand schedule of the form

$$n = \varphi(w)$$

can be derived from the optimization problem. Explain how you characterize the function φ. What is the sign of $\varphi'(w)$? Why?

Solution: The firm's problem is to maximize a strictly concave function. The necessary and sufficient first order condition is

$$f'(n) = w$$

For each w, this implicitly determines a unique $n = \varphi(w)$. (Well defined since $f'(n)$ is strictly decreasing in n). Write this implicit function as

$$f'[\varphi(w)] = w$$

So on differentiating with respect to w

$$f''[\varphi(w)]\varphi'(w) = 1$$

or

$$\varphi'(w) = \frac{1}{f''[\varphi(w)]} < 0$$

When an additional unit of labor produces less at the margin, a rise in the wage rate will reduce the firm's demand for labor.

(b) (5 marks): Explain the stochastic dynamics that n_t exhibits. Carefully explain how you could simulate the optimal labor demand choices.

Solution: Clearly $n_t = \varphi(w_t)$ inherits the stochastic dynamics of the Markov chain for w_t (since $n_t = \varphi(w_t)$ is monotonic). We can say that n_t itself follows a Markov chain, namely $(\varphi(w), P, \mu_0)$. We could simulate $\{n_t\}$ by simulating $\{w_t\}$ and for each t calculate $n_t = \varphi(w_t)$. In particular, we use the initial distribution μ_0 to determine initial wages and employment, and then use the transition matrix P (and a random number generator) to iterate forward.

(c) (10 marks): Suppose that the production function is

$$f(n) \equiv n^{\gamma}, \qquad 0 < \gamma < 1$$

Provide an explicit solution for $n = \varphi(w)$ and for profits $\pi(w)$. What pattern of labor supply would one observe given the fluctuations in w? How does your answer depend on the labor share γ? What is the elasticity of labor demand?

Solution: The first order condition can be written

$$\gamma n^{\gamma-1} = w$$

or

$$\begin{aligned} \log(n) &= -\frac{1}{1-\gamma}[\log(w) - \log(\gamma)] \\ n &= \left(\frac{w}{\gamma}\right)^{-\frac{1}{1-\gamma}} \equiv \varphi(w) \end{aligned}$$

The elasticity of labor demand is

$$\frac{d\log(n)}{d\log(w)} = -\frac{1}{1-\gamma} < 0$$

Plugging the solution for labor demand back into the objective function, we have

$$\begin{aligned} \pi(w) &= f[\varphi(w)] - w\varphi(w) \\ &= \left(\frac{w}{\gamma}\right)^{-\frac{\gamma}{1-\gamma}} - w\left(\frac{w}{\gamma}\right)^{-\frac{1}{1-\gamma}} \end{aligned}$$

We can easily deduce the following relationships between log wages and log labor demand

$$\mathsf{E}\{\log(n_t)\} = -\frac{1}{1-\gamma}[\mathsf{E}\{\log(w_t)\} - \log(\gamma)]$$

and

$$\mathsf{Var}\{\log(n_t)\} = \frac{1}{(1-\gamma)^2}\mathsf{Var}\{\log(w_t)\}$$

Hence if γ is large ($\gamma \to 1$) we should see low average employment. Also, since $0 < \gamma < 1$ we should see that employment is less volatile than wages.

(d) (5 marks). Let the production function be as in part (c). Suppose that the Markov chain has $m = 2$ states with

$$w = \begin{pmatrix} w_L & w_H \end{pmatrix}$$

and transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \qquad 0 < p, q < 1$$

Finally, the initial distribution is

$$\mu_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Solve for the stationary distribution of wages. Explain how the mean and variance of the stationary distribution of wages depends on the transition probabilities p and q.

Solution: This Markov chain has a unique invariant distribution $\bar{\pi}$ which solves

$$0 = (I - P')\bar{\pi}$$

or

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p & -q \\ -p & q \end{pmatrix} \begin{pmatrix} \bar{\pi}_L \\ \bar{\pi}_H \end{pmatrix}$$

Carrying out the calculations, we see that

$$\bar{\pi}_L = \frac{q}{p}\bar{\pi}_H$$

But we also know that these elements must satisfy

$$\bar{\pi}_L + \bar{\pi}_H = 1$$

So we can solve these two equations in two unknowns to get

$$\bar{\pi}_L = \frac{q}{p+q}, \qquad \bar{\pi}_H = \frac{p}{p+q}$$

As $q \to 0$, the state w_L becomes a transient state and the state w_H becomes an absorbing state: once the chain leaves w_L, it never returns. Since this happens with probability 1 if we run the chain long enough, the stationary distribution will be degenerate with $\bar{\pi} \to (0, 1)$. Similarly, as $p \to 0$, the state w_H becomes a transient state and the state w_L becomes an absorbing state and $\bar{\pi} \to (1, 0)$.

(e) (5 marks). Suppose further that we have $p = q$ and $w_L = \omega - 1$, $w_H = \omega + 1$ for some $\omega > 1$. Compute the mean and variance of the implied stationary distribution of labor demand. Explain how your answers depend on the parameters ω and γ. Give economic intuition.

Solution: If $p = q$ the stationary distribution is just

$$\begin{pmatrix} \bar{\pi}_L \\ \bar{\pi}_H \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

So the stationary mean is just

$$\begin{aligned} \mathsf{E}\{\varphi(w)\} &= \sum_i \varphi(w_i)\bar{\pi}_i \\ &= \frac{1}{2}\varphi(\omega - 1) + \frac{1}{2}\varphi(\omega + 1) \\ &= \frac{1}{2}\left(\frac{\omega - 1}{\gamma}\right)^{-\frac{1}{1-\gamma}} + \frac{1}{2}\left(\frac{\omega + 1}{\gamma}\right)^{-\frac{1}{1-\gamma}} \end{aligned}$$

Notice that because $\varphi(w)$ is convex,

$$\mathsf{E}\{\varphi(w)\} > \varphi(\mathsf{E}\{w\}) = \varphi(\omega)$$

The stationary variance is

$$\mathsf{Var}\{\varphi(w)\} = \mathsf{E}\{\varphi(w)^2\} - \mathsf{E}\{\varphi(w)\}^2$$

The second moment is

$$\begin{aligned} \mathsf{E}\{\varphi(w)^2\} &= \sum_i \varphi(w_i)^2 \bar{\pi}_i \\ &= \frac{1}{2}\varphi(\omega-1)^2 + \frac{1}{2}\varphi(\omega+1)^2 \\ &= \frac{1}{2}\left(\frac{\omega-1}{\gamma}\right)^{-\frac{2}{1-\gamma}} + \frac{1}{2}\left(\frac{\omega+1}{\gamma}\right)^{-\frac{2}{1-\gamma}} \end{aligned}$$

and we can now calculate the variance (the second centered moment) by subtracting $\mathsf{E}\{\varphi(w)\}^2$. I did not expect you to grind out the algebra of these expressions, only to provide the general formulas, especially to recognize that you computed the mean labor supply as $\mathsf{E}\{n\} = \mathsf{E}\{\varphi(w)\} = \sum_i \varphi(w_i)\bar{\pi}_i$.

Question 3. *Cake Eating* (30 marks): Consider a consumer with utility function

$$\sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0 < \beta < 1$$

The consumer is endowed with a cake of size x_0 at time $t = 0$. Each period, she has cake x_t and can either consume some, c_t, or hold some cake over to next period, x_{t+1}.

(a) (10 marks): Provide a dynamic programming representation of this problem. In your answer, let $V(x)$ denote the utility value of a cake of size x.

Solution: The Bellman equation for this problem is

$$V(x) = \max_{x' \geq 0} \{U(c) + \beta V(x')\}$$

where the maximization is subject to the constraint

$$c + x' \leq x$$

(b) (15 marks): Let the period utility function be

$$U(c) \equiv \frac{c^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0$$

Guess that the value function $V(x)$ and policy function $g(x)$ for your dynamic programming problem have the forms

$$\begin{aligned} V(x) &= \alpha \frac{x^{1-\sigma}}{1-\sigma} \\ g(x) &= \theta x \end{aligned}$$

for some unknown coefficients $\alpha > 0$ and $0 < \theta < 1$. Solve for the unknown coefficients.

Solution: The first order and envelope conditions for this problem are

$$U'(x - x') = \beta V'(x')$$

and

$$V'(x) = U'(c)$$

Using the budget constraint $c = x - x'$ and the guess for the policy function $x' = \theta x$ we have $c = (1-\theta)x$ and so the Euler equation can be written

$$U'[(1-\theta)x] = \beta U'[(1-\theta)x\theta]$$

With the assumed utility function, $U'(c) = c^{-\sigma}$ and so

$$[(1-\theta)x]^{-\sigma} = \beta[(1-\theta)x\theta]^{-\sigma}$$

Simplifying, this gives

$$\theta = \beta^{\frac{1}{\sigma}}$$

(Clearly, $0 < \beta^{\frac{1}{\sigma}} < 1$). Now from the envelope condition

$$V'(x) = U'[(1-\theta)x]$$

and so using the guess for the value function, we have

$$\alpha x^{-\sigma} = [(1-\theta)x]^{-\sigma}$$

Simplifying gives

$$\begin{aligned} \alpha &= (1-\theta)^{-\sigma} \\ &= (1-\beta^{\frac{1}{\sigma}})^{-\sigma} > 0 \end{aligned}$$

(c) (5 marks): Verify that over the infinite time horizon, the consumer eats all the cake, namely

$$\sum_{t=0}^{\infty} c_t = x_0$$

At what rate does the cake diminish? Provide a formula that calculates how long it takes for there to only be $\varepsilon > 0$ crumbs of cake left. How does this rate of consumption depend on the parameters β and σ? Give economic intuition.

Solution: Clearly,

$$c_t = (1-\theta)x_t$$

and

$$x_{t+1} = \theta x_t, \qquad x_0 \text{ given}$$

So solving the difference equation gives the consumption function

$$c_t = (1-\theta)\theta^t x_0$$

and over time

$$\sum_{t=0}^{\infty} c_t = \sum_{t=0}^{\infty} (1-\theta)\theta^t x_0 = (1-\theta)x_0 \sum_{t=0}^{\infty} \theta^t = \frac{(1-\theta)}{(1-\theta)} x_0 = x_0$$

(which is well-defined since $\theta = \beta^{\frac{1}{\sigma}}$ and $0 < \beta^{\frac{1}{\sigma}} < 1$). The cake diminishes geometrically at rate $\beta^{\frac{1}{\sigma}} - 1$. The more patient the consumer (higher β) the more slowly the cake diminishes. As $\sigma \to 0$, the utility function becomes nearly linear in consumption and consumption at different dates are perfect substitutes (modulo β). So as $\sigma \to 0$, the consumer gorges all the cake in a binge at date $t = 0$. As $\sigma \to \infty$, consumption at different dates are perfect complements and the consumer tries to have a very flat consumption profile over time. Now call $T > 0$ the date at which there are $\varepsilon > 0$ crumbs left. This date solves the equation

$$\varepsilon = \theta^T x_0$$

Taking logs and rearranging gives

$$T = \frac{\log(\frac{\varepsilon}{x_0})}{\log(\theta)} = \frac{\log(\frac{\varepsilon}{x_0})}{\log(\beta^{\frac{1}{\sigma}})} = \sigma \frac{\log(\frac{\varepsilon}{x_0})}{\log(\beta)}$$

Although $\log(\beta) < 0$ since $\beta < 1$, we also have $\log(\frac{\varepsilon}{x_0}) < 0$ since $\varepsilon < x_0$. So $T \geq 0$. Notice that $T \to 0$ as $\sigma \to 0$ and $T \to \infty$ as $\sigma \to \infty$, in line with the perfect substitutes/complements discussion above.