316-406 Advanced Macroeconomic Techniques

Problem Set #2 *Due 1 September, 2004*

For this problem set you'll probably have to use a more sophisticated programming language, and not just Excel. Go on, give Matlab (or Gauss) a whirl!

Question 1. (*Stochastic difference equations*). Consider the linear stochastic difference equation

$$X_{t+1} - aX_t = b + \sigma Z_{t+1}$$

with given parameters a, b, σ and given initial condition $X_0 = x_0$. Let the innovations be IID standard normal, $Z_{t+1} \sim N(0, 1)$.

- Fix $x_0 = 0$ and $\sigma = 1$ and simulate paths of length $T = 100$ and $T = 500$ for each of $a = 0.5$ and $a = 0.99$ and for each of $b = 0$ and $b = 1$.
- For each simulation (there are 6) plot the sample path, and lines indicating the long-run mean of $\{X_t\}$ and the long run mean $\pm$ 2 standard deviations.
- Put all these plots on a single page (e.g., in Matlab use the `subplot(3,2,i)` command for `i` $= 1, \ldots 6$).
- To draw random numbers from the $N(0, 1)$, Matlab has the command `randn()`.
- Now for the $T = 500$ simulations, drop the first 400 observations and compute a histogram for the remaining 100 observations (e.g., in Matlab use the `hist()` command), and graph on the same plot the density function for the long run distribution of $\{X_t\}$ for each parameter configuration. Briefly explain your findings.

Question 2. (*Markov chains*). Consider a symmetric, two-state Markov chain (x, P, π_0) with state space

$$x = (\mu + \sigma, \mu - \sigma)$$

for given μ (which may be positive or negative) and $\sigma > 0$. Let the transition probabilities be

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

for $0 < p < 1$.

- Compute the unique stationary distribution of this Markov chain.
- Use this stationary distribution to show that the long run distribution of X_t has mean

  $$\mathsf{E}\{X_t\} = \mu$$

  and standard deviation

  $$\mathsf{Std}\{X_t\} = \sigma$$

- What property of the process $\{X_t\}$ does the parameter p control? How does this property depend on whether p is greater or less than 0.5?

- Now set $\pi_0 = (1, 0)$ so that the chain starts in state $x_1 = \mu + \sigma$ with probability 1. Let $\mu = 0.02$ and let $\sigma = 0.04$. Using Matlab or a similar program, compute **sample paths** $\{x_t\}_{t=1}^T$ of length $T = 50, 100$, and 1000 for the each of $p = 0.1, 0.5$, and 0.9. For each run, compute the following three sample statistics

$$\begin{aligned}
\bar{x} &= \frac{1}{T}\sum_{t=1}^{T} x_t \\
s^2 &= \frac{1}{T-1}\sum_{t=1}^{T}(x_t - \bar{x})^2 \\
\rho &= \frac{\sum_{t=2}^{T}(x_t - \bar{x})(x_{t-1} - \bar{x})}{\sum_{t=1}^{T}(x_t - \bar{x})^2}
\end{aligned}$$

(Note: you will have to compute $3 \times 3 \times 3 = 27$ numbers). How close do your sample statistics get to their population counterparts? How does this depend on the length of T and the magnitude of p?

Question 3. (*Preferences over Markov chains*). Let a consumer have preferences

$$\mathsf{E}\left\{\sum_{t=0}^{\infty}\beta^t \frac{C_t^{1-\gamma}}{1-\gamma}\right\}$$

where $0 < \beta < 1$ and $\gamma > 1$. Let consumption $\{C_t\}$ follow a Markov chain (c, P, π_0) with $c = [c_i]$ an n-vector. Define vectors $u = [u_i]$, $v = [v_i]$ where

$$\begin{aligned}
u_i &\equiv \frac{c_i^{1-\gamma}}{1-\gamma} \\
v_i &\equiv \mathsf{E}_i\left\{\sum_{t=0}^{\infty}\beta^t \frac{C_t^{1-\gamma}}{1-\gamma}\right\} \equiv \mathsf{E}\left\{\sum_{t=0}^{\infty}\beta^t \frac{C_t^{1-\gamma}}{1-\gamma}\,\middle|\, C_0 = c_i\right\}
\end{aligned}$$

and define a scalar

$$V = \mathsf{E}\{v_i\}$$

- Show that the vector v is given by

$$v = (I - \beta P)^{-1} u$$

and that expected utility is therefore

$$V = \sum_{i=1}^{n} v_i \pi_{0,i}$$

- Now let $c = (1, 5)$ and $\pi_0 = (0.5, 0.5)$ and consider Markov chains with transition probabilities

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

Let $\beta = 0.95$ and $\gamma = 2.5$. Compute V. Which Markov chain does the consumer prefer? Redo the calculation with $\gamma = 4$. Now which Markov chain does the consumer prefer? Explain why your answer does or does not change. [This follows Ljungqvist and Sargent, Exercise 1.3].

Question 4. (*Yet more on Markov chains*). Consider a model where GDP growth $\{G_t\}$ follows a two-state Markov chain (g, Q, π_0^g) with states

$$g = (g_H, g_L) = (\mu + \sigma, \mu - \sigma) = (0.06, -0.02)$$

transition probabilities

$$Q = \begin{pmatrix} 0.99 & 0.01 \\ 0.25 & 0.75 \end{pmatrix}$$

and initial distribution $\pi_0^g = (1, 0)$.

Now suppose that an individual's employment $\{E_t\}$ follows a Markov chain (e, P_g, π_0^e) with employment states

$$e = (e_H, e_L) = (0, 1)$$

and **state-dependent** transition probabilities

$$\begin{aligned} P_H &= \begin{pmatrix} 0.99 & 0.01 \\ 0.9 & 0.1 \end{pmatrix}, \qquad \text{if } g = g_H \\ P_L &= \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix}, \qquad \text{if } g = g_L \end{aligned}$$

and initial distribution $\pi_0^e = (1, 0)$.

- Show how to represent this coupled system of Markov chains in terms of a single state variable x, transition matrix P_x and initial distribution π_0^x.
- Suppose that a worker gets wage $w = 0.10$ if $e = 1$ and $w = 0.08$ if $e = 0$. Simulate a Markov chain for the individual's wage process $\{W_t\}_{t=1}^T$ for $T = 100$. Repeat this simulation 4 times and plot the sample paths on a page with 4 subplots. In each subplot, also show the associated GDP growth rates. Interpret the patterns you observe in this artificial data. In the long run, what fraction of time does the economy spend in recession ($g = g_L$)? What fraction of time does a person spend in unemployment ($e = e_L$)?
- Repeat this exercise with the following transition matrices

$$Q = \begin{pmatrix} 0.99 & 0.01 \\ 0.01 & 0.99 \end{pmatrix}$$

$$\begin{aligned} P_H &= \begin{pmatrix} 0.99 & 0.01 \\ 0.99 & 0.01 \end{pmatrix} \qquad \text{if } g = g_H \\ P_L &= \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \qquad \text{if } g = g_L \end{aligned}$$

  What kind of behavior does the wage process $\{W_t\}$ seem to exhibit? Does this seem intuitive?

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