316-406 ADVANCED MACROECONOMIC TECHNIQUES NOTE 7b

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Aiyagari's model

Arguably the most popular example of a simple incomplete markets model is due to Rao Aiyagari (1994, QJE). As in Huggett's paper, market incompleteness and idiosyncratic shocks give rise to **endogenous heterogeneity**. Essentially, Aiyagari's presents a version of the neoclassical growth model with idiosyncratic but no aggregate risk.

A. Households

Households face constant factor prices (w, r). The household's labor opportunities n arrive according to a Markov chain with transition probabilities $\pi(n', n)$ where

$$\pi(n', n) = \Pr(n_{t+1} = n' | n_t = n)$$

and where $n, n' \in \mathcal{N}$, a finite set. For example, if there are two states we might think of the household as being either employed ($n = 1$) or unemployed ($n = 0$). Households accumulate physical capital k which rents for r and depreciates at δ. The household budget constraint is therefore

$$c + k' \leq (1 + r - \delta)k + wn$$

The dynamic programming problem of a household with capital k and labor opportunities n is represented by

$$V(k, n) = \max_{k' \geq 0} \left\{ U(c) + \beta \mathsf{E}[V(k', n')|n] \right\}$$

Households cannot borrow to insure themselves against idiosyncratic shocks, but they can **self-insure** by holding physical capital. (It is easy to modify the model so that households can also borrow, but only up to some finite limit).

As usual, when solving this dynamic programming problem we will restrict k' to belong to a grid of the form

$$k' \in \mathcal{K} \equiv [0 < \cdots < k_{\max}]$$

where $k_{\max}$ is a non-binding upper limit. We will then solve for a value function $V(k, n)$ and a policy function $k' = g(k, n)$ on the discrete state space $\mathcal{K} \times \mathcal{N}$. As in Huggett's model, the value and policy functions depend on the constant factor prices (w, r) and when we solve for an equilibrium we will have to solve for constant (w, r) consistent with optimization by households (and firms).

The exogenous Markov process for n, n' and the policy function $k' = g(k, n)$ together with initial conditions k_0, n_0 determine an **endogenous** Markov chain for (k, n) pairs on the state space $\mathcal{K} \times \mathcal{N}$. Under mild regularity conditions, this "big" Markov chain has a **stationary distribution** that we will denote by $\mu(k, n)$, that is

$$\mu(k, n) = \Pr(k_t = k, n_t = n)$$

As with Huggett's model, this stationary distribution has both a time-series and a cross-section interpretation.

B. Aggregates

Let K and N denote the average per capita ("aggregate") physical capital stock and level of employment. In a stationary equilibrium, to be defined below, these will be given by

$$K = \sum_{k \in \mathcal{K}} \sum_{n \in \mathcal{N}} g(k, n)\mu(k, n)$$

and

$$N = \sum_{n \in \mathcal{N}} n\bar{\pi}(n)$$

where $\bar{\pi}$ denotes the stationary distribution of the exogenous Markov chain for employment states. In this model, aggregate employment is fixed exogenously (it is determined exclusively by the properties of the Markov chain for employment states) but aggregate capital is determined endogenously and depends on the entire distribution of (k, n) pairs in the population.

C. Firms

In this model, firms are all identical so we can think of there being a single representative firm that takes factor prices (w, r) as given and that chooses capital and labor input each period to maximize

$$F(K, N) - rK - wN$$

where $F(K,N)$ is the firm's production function. The optimal choices of capital and labor are then given by the first order conditions

$$\begin{aligned} \frac{\partial F(K,N)}{\partial K} &= r \\ \frac{\partial F(K,N)}{\partial N} &= w \end{aligned}$$

For example, with the usual constant returns Cobb-Douglas production function, $F(K,N) = K^\alpha N^{1-\alpha}$, we would have

$$\begin{aligned} \alpha \left(\frac{K}{N}\right)^{\alpha-1} &= r \\ (1-\alpha)\left(\frac{K}{N}\right)^{\alpha} &= w \end{aligned}$$

D. Equilibrium concept

DEFINITION. A **stationary recursive competitive equilibrium** for this economy is: (i) a value function V, (ii) an individual decision rule g, (iii) a stationary probability distribution μ, (iv) factor prices (w,r), and (v) aggregate capital and employment (K,N) such that:

1. Given the factor prices (w,r), the value function V and the individual decision rule g solve the household's dynamic programming problem,
2. Given the factor prices (w,r), the aggregate capital and employment (K,N) solve the firm's static optimization problem,
3. The stationary distribution μ is induced via the exogenous Markov chain for n, n' and the policy function g, and
4. The aggregate capital stock is implied by the aggregate of household decisions

$$K = \sum_{k\in\mathcal{K}}\sum_{n\in\mathcal{N}} g(k,n)\mu(k,n)$$

and aggregate employment is

$$N = \sum_{n\in\mathcal{N}} n\bar{\pi}(n)$$

E. Computing an equilibrium

In practice, we use an algorithm of the following kind:

1. Use the properties of the Markov Chain for n, n' to solve for the exogenous N.
2. Guess an initial level of the stationary aggregate capital stock K_0.
3. Use (K_0, N) to solve the firm's marginal productivity conditions for the implied factor prices (w_0, r_0).
4. Taking the factor prices (w_0, r_0) as given solve the household's dynamic programming problem to obtain the policy function $g_0(k, n)$.
5. Using that policy function and the Markov chain for employment states, construct the transition matrix on the state space $\mathcal{K} \times \mathcal{N}$ and solve for the stationary probability distribution $\mu_0(k, n)$ over $\mathcal{K} \times \mathcal{N}$.
6. Compute the implied aggregate capital

$$K_0^* = \sum_{k \in \mathcal{K}} \sum_{n \in \mathcal{N}} g_0(k, n) \mu_0(k, n)$$

   Test whether $\|K_0^* - K_0\| < \varepsilon$ for some small ε, say $\varepsilon = 10^{-6}$. If the error is greater than ε, update to a new guess for the aggregate capital stock by setting

$$K_1 = \xi K_0 + (1 - \xi) K_0^*$$

   for some value $0 < \xi < 1$. This "relaxation" parameter is often set to a number like $\xi = 0.10$.
7. Keep iterating on this scheme with

$$K_{j+1} = \xi K_j + (1 - \xi) K_j^*$$

   for $j = 0, 1, ...$ until the convergence criterion $\left\| K_j^* - K_j \right\| < \varepsilon$ is met.

In an equilibrium of this kind, individual capital stocks and consumption are **stochastic processes**

$$\begin{aligned} k_{t+1} &= g(k_t, n_t) \\ c_t &= (1 + r - \delta) k_t + w n_t - k_{t+1} \end{aligned}$$

where n_t follows an exogenous Markov chain. Individual capital and consumption fluctuate, but the distribution of such capital and consumption positions across households is constant. At any date, the **fraction** of households with capital k and employment n is constant and given by $\mu(k, n)$.

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