# 316-406 ADVANCED MACROECONOMIC TECHNIQUES NOTE 4a

*Chris Edmond* ⟨cpedmond@unimelb.edu.au⟩

In this course, we will consider two ways to model uncertainty. We will adopt the following notational conventions: **random variables** will be denoted by capital letters, like $X_t$ and $Z_t$, **realizations** of random variables will be denoted by corresponding little letters, say $x_t$ and $z_t$, a **stochastic process** will be a sequence of random variables, say $\{X_t\}$ and $\{Z_t\}$, and a **sample path** will be a sequence of realizations, say $\{x_t\}$ and $\{z_t\}$. If a random variable $X_t$ has realizations that take values in some continuous set, we will model the process $\{X_t\}$ in terms of **stochastic difference equations**. If $X_t$ has realizations that take values in some discrete set, we will model the process $\{X_t\}$ in terms of a **Markov chain**.

## Stochastic difference equations

We will frequently want to solve a linear stochastic difference equation of the form

$$X_{t+1} - aX_t = b + \sigma Z_{t+1}, \qquad t = 0, 1, 2, ... \tag{1}$$

given scalars $a, b$ and $\sigma \geq 0$, an **initial realization** $X_0 = x_0$ and an exogenous stochastic process $\{Z_t\}$. If $\sigma = 0$, then we have as a special case the deterministic difference equation $X_{t+1} - aX_t = b$. Otherwise, if $\sigma > 0$ we have a difference equation that is buffeted by shocks. With this model the primitive stochastic process is $\{Z_t\}$ which **induces** a new process $\{X_t\}$ according to the rule in (1). In econometrics, a linear stochastic difference equation is sometimes also known as an autoregression and the particular equation in (1) is known as an AR(1).

We will typically assume that each random variable $Z_t$ is an independent **standard normal** so that

$$\mathsf{E}\{Z_t\} = 0, \qquad \mathsf{V}\{Z_t\} = 1, \qquad t = 0, 1, 2, ...$$

and

$$\mathsf{E}\{Z_t Z_{t+k}\} = 0, \qquad t, k = 0, 1, 2, ...$$

The main reason for assuming normal shocks is that linear combinations of normal random variables are also normal random variables, so if the $Z_t$ are normal, the induced random variables $X_t$ will also be normal. To see how useful this is, recall that if $Z$ is standard normal, then $X = \mu + \sigma Z$ is also normal with mean $\mu$ and standard deviation $\sigma$.

So, **conditional** on $X_0 = x_0$, the random variable $X_1$ is given by

$$X_1 = b + ax_0 + \sigma Z_1$$

with mean and variance

$$\begin{aligned} \mathsf{E}_0\{X_1\} &\equiv \mathsf{E}\{X_1|X_0 = x_0\} = b + ax_0 \\ \mathsf{V}_0\{X_1\} &\equiv \mathsf{V}\{X_1|X_0 = x_0\} = \sigma^2 \end{aligned}$$

Another notation for this is that if $Z_1$ is standard normal [written $Z_1 \sim \mathcal{N}(0,1)$], then the **conditional distribution** for $X_1|x_0$ is also normal with

$$X_1|x_0 \sim \mathcal{N}(b + ax_0, \sigma^2)$$

And more generally,

$$X_{t+1}|x_t \sim \mathcal{N}(b + ax_t, \sigma^2)$$

If $|a| < 1$, we can solve a linear stochastic difference equation by iterating in the usual manner, namely

$$X_1 = b + a^1 x_0 + \sigma Z_1$$

and for $X_2$

$$\begin{aligned} X_2 &= b + a^1 X_1 + \sigma Z_2 \\ &= b + a^1(b + a^1 x_0 + \sigma Z_1) + \sigma Z_2 \\ &= b + a^1 b + a^2 x_0 + \sigma a^1 Z_1 + \sigma Z_2 \end{aligned}$$

and for $X_3$

$$\begin{aligned} X_3 &= b + a^1 X_2 + \sigma Z_3 \\ &= b + a^1(b + a^1 b + a^2 x_0 + \sigma a^1 Z_1 + \sigma Z_2) + \sigma Z_3 \\ &= b + a^1 b + a^2 b + a^3 x_0 + \sigma a^2 Z_1 + \sigma a^1 Z_2 + \sigma Z_3 \end{aligned}$$

and for general $X_t$,

$$\begin{aligned} X_t &= a^t x_0 + \sum_{i=0}^{t-1} a^i b + \sum_{i=0}^{t-1} a^i \sigma Z_{t-i} \\ &= a^t x_0 + \frac{1-a^t}{1-a} b + \sigma \sum_{i=0}^{t-1} a^i Z_{t-i} \end{aligned}$$

Since $X_t$ is a linear combination of normal random variables, it is also normal with mean

$$\begin{aligned} \mathsf{E}_0\{X_t\} &= a^t x_0 + \frac{1-a^t}{1-a} b + \sigma \mathsf{E}_0 \left\{ \sum_{i=0}^{t-1} a^i Z_{t-i} \right\} \\ &= a^t x_0 + \frac{1-a^t}{1-a} b + \sigma \sum_{i=0}^{t-1} a^i \mathsf{E}_0 \{Z_{t-i}\} \\ &= a^t x_0 + \frac{1-a^t}{1-a} b \end{aligned}$$

and variance

$$\begin{aligned} \mathsf{V}_0\{X_t\} &= \mathsf{V}_0 \left\{ \sigma \sum_{i=0}^{t-1} a^i Z_{t-i} \right\} \\ &= \sigma^2 \sum_{i=0}^{t-1} a^{2i} \mathsf{V}_0\{Z_{t-i}\} \\ &= \frac{1-a^{2t}}{1-a^2} \sigma^2 \end{aligned}$$

(This calculation uses the fact that the shocks $Z_t$ are independent and so have zero covariances. Because of this, the variance of the sum is just the sum of the variances).

In short, the distribution of $X_t$ is normal with

$$X_t | x_0 \sim \mathcal{N}\left( a^t x_0 + \frac{1-a^t}{1-a} b, \frac{1-a^{2t}}{1-a^2} \sigma^2 \right)$$

This is the distribution of $X_t$ conditional only the trivial initial realization.

### Stationary distributions

Now recall the usual stability criteria for deterministic difference equations: if $|a| < 1$, then a linear deterministic difference equation is globally stable and converges to a unique steady state, a number $\bar{x}$. With a stochastic difference equation we have no hope of finding a single steady state. Instead, we look for a steady state distribution or **stationary distribution** of $X_t | x_0$.

Taking the limit as $t \to \infty$, we see that

$$\lim_{t\to\infty} \mathsf{E}_0\{X_t\} = \frac{b}{1-a}$$

and

$$\lim_{t\to\infty} \mathsf{V}_0\{X_t\} = \frac{\sigma^2}{1-a^2}$$

So as we iterate, the dependence of the distribution on the initial realization $x_0$ vanishes and the distribution of $X_t$ settles down to a normal distribution with a mean equal to the steady state of the deterministic difference equation (the "non-stochastic steady state") and a variance that depends on both the variance of the shocks $\sigma^2$, and the persistence of the difference equation as measured by $a$.

Two special cases are worthy of further comment.

### A. IID case

If $a = 0$, then the difference equation reduces to

$$X_{t+1} = b + \sigma Z_{t+1}, \qquad t = 0, 1, 2, ...$$

In this case, we say that $\{X_t\}$ is **independent and identically distributed** (IID). The distribution of $X_t|x_0$ for $t > 0$ does not depend on $x_0$ and is just

$$X_t|x_0 \sim \mathcal{N}(b, \sigma^2)$$

If $\{X_t\}$ is IID, then there is no persistence.

### B. Random walk case

If $a = 1$, then the difference equation reduces to

$$X_{t+1} - X_t = b + \sigma Z_{t+1}, \qquad t = 0, 1, 2, ...$$

In this case, we say that $\{X_t\}$ is a **random walk**. If $b \neq 0$ then it is a random walk with drift. If $\{X_t\}$ is a random walk, its **differences** have a well-behaved distribution,because the difference

$\Delta X_{t+1} = X_{t+1} - X_t$ is IID. So

$$\begin{aligned} \mathsf{E}_t\{\Delta X_{t+1}\} &= b \\ \mathsf{V}_t\{\Delta X_{t+1}\} &= \sigma^2 \end{aligned}$$

If $b \neq 0$, then $\{X_t\}$ trends up or down over time with a constant expected change of $\mathsf{E}_t\{\Delta X_{t+1}\} = b$.

We will see however, that if $\{X_t\}$ is a random walk its **levels** do not have a well-behaved stationary distribution. Iterating in the usual manner gives

$$X_1 = b + x_0 + \sigma Z_1$$

and for $X_2$

$$\begin{aligned} X_2 &= b + X_1 + \sigma Z_2 \\ &= b + (b + x_0 + \sigma Z_1) + \sigma Z_2 \\ &= 2b + x_0 + \sigma Z_1 + \sigma Z_2 \end{aligned}$$

and for general $X_t$,

$$X_t = x_0 + tb + \sum_{i=0}^{t-1} \sigma Z_{t-i}$$

So

$$\mathsf{E}_0\{X_t\} = x_0 + tb$$

and the limit of $\mathsf{E}_0\{X_t\}$ as $t \to \infty$ diverges to $\pm\infty$ depending on the sign of $b$. Similarly,

$$\mathsf{V}_0\{X_t\} = t\sigma^2$$

so the limit of $\mathsf{V}_0\{X_t\}$ as $t \to \infty$ is $+\infty$.

In short, there is a close relationship between stability for a deterministic difference equation and the existence of a stationary distribution for the associated stochastic difference equation.

*Chris Edmond*

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