316-406 ADVANCED MACROECONOMIC TECHNIQUES NOTE 2b

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Systems of linear difference equations

We will frequently want to solve a two-dimensional system of difference equations

$$x_{t+1} - Ax_t = b, \qquad t \geq 0$$

given a 2-by-2 matrix A, a 2-by-1 vector b and a 2-by-1 vector of initial conditions x_0. (Higher dimensional systems generalize in an obvious way, so I will do almost everyting here for the two dimensional case).

A. Diagonal coefficient matrix

If the matrix A is **diagonal**, i.e., of the form

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$$

then there is no interaction between the two equations (they are independent) and can be written

$$\begin{aligned} x_{1,t+1} - a_{11}x_{1,t} &= b_1 \\ x_{2,t+1} - a_{22}x_{2,t} &= b_2 \end{aligned}$$

These have the obvious solutions

$$\begin{aligned} x_{1,t} &= (1 - a_{11}^t)\bar{x}_1 + a_{11}^t x_{1,0} \\ x_{2,t} &= (1 - a_{22}^t)\bar{x}_2 + a_{22}^t x_{2,0} \end{aligned}$$

or in matrix form

$$x_t = (I - A^t)\bar{x} + A^t x_0$$

where I is a conformable identity matrix. In this expression, the matrix power A^t is justified because A is diagonal so that

$$A^t = \begin{pmatrix} a_{11}^t & 0 \\ 0 & a_{22}^t \end{pmatrix}$$

(A^t does not equal each of its elements to the power t unless A is diagonal). The steady state vector $\bar{x}$ is the natural analogue to the scalar case,

$$\bar{x} = (I - A)^{-1} b$$

The stability of the system depends on the behavior of each component. For example, suppose we had the diagonal system

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 2.0 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$

with solution

$$\begin{aligned} x_{1,t} &= 0.5^t x_{1,0} \\ x_{2,t} &= 2.0^t x_{2,0} \end{aligned}$$

Then the system is unstable because one component converges to zero but the other explodes. Finally notice that with a system of difference equations, we need as many "initial conditions" as there are endogenous variables in order to pin down the solution completely.

Coupled equations

If the square matrix A is not diagonal, there is feedback between the two equations and we can no longer solve them independently. In this case, the equations are said to be **coupled**. Since solving diagonal system is easy, it would be nice if there was a change of variables that allowed us to "uncouple" or "diagonalize" the system. In fact, for an important class of matrices, we can make this change of variables quite easily.

A square matrix A can be **diagonalized** if there exists an invertible matrix Q such that the matrix D defined by

$$D = Q^{-1} A Q$$

is diagonal. Then if A can be diagonalized, we can use the change of variables $z_t = Q^{-1} x_t$ to write the system of difference equations as

$$x_{t+1} - A x_t = b \Longleftrightarrow Q z_{t+1} - A Q z_t = b$$

or

$$z_{t+1} - Q^{-1}AQz_t = z_{t+1} - Dz_t = Q^{-1}b$$

which has the solution

$$\begin{aligned} z_t &= (I - D^t)\bar{z} + D^t z_0 \\ \bar{z} &= (I - D)^{-1}Q^{-1}b \end{aligned}$$

and we can always recover the orignal x_t variables by plugging back in the definition $x_t = Qz_t$. Now that we know why matrices that can be diagonalized are going to be useful, it's time to brush up on some important concepts from linear algebra that will actually help us do diagonalizations. We'll draw all the concepts together as we near the end of this note.

If A is a square matrix, a scalar λ is an **eigenvalue** of A if and only if $A - \lambda I$ is a singular matrix. A square matrix is **singular** if and only if its **determinant** is zero. Hence λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = 0$$

For a scalar b, the determinant is $\det(b) = b$, so that if (say) $b = a - \lambda$ is scalar its only eigenvalue is the number λ such that $a - \lambda = 0$ or $\lambda = a$. If the matrix has more than one dimension, it will typically have more than one eigenvalue. For example, if B is 2-by-2, the determinant is

$$\det(B) = \det\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = b_{11}b_{22} - b_{12}b_{21}$$

Determinants of higher order matrices are calculated in similar fashion and are easy to compute with a matrix programming language like Matlab. Using $B = A - \lambda I$, the definition of the determinant for a 2-by-2 system gives

$$\begin{aligned} \det(A - \lambda I) &= \det\begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \\ &\equiv p(\lambda) \end{aligned}$$

which is known as the **characteristic polynomial** in λ. For a 2-by-2 system, this is just a quadratic equation (higher dimensional A lead to higher dimensional polynomials, but that need not bother us here). That is, finding the eigenvalues of A just means finding the roots of

$$p(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

or by the quadratic formula

$$\lambda_1, \lambda_2 = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

where

$$\begin{aligned} \alpha &\equiv 1 \\ \beta &\equiv -(a_{11} + a_{22}) \\ \gamma &\equiv (a_{11}a_{22} - a_{12}a_{21}) \end{aligned}$$

Of course the roots may be real or complex. Generally, a square matrix has as many eigenvalues as its dimension, but some of these values may be **repeated**. For example, in the 2-by-2 case, if the **discriminant** $\beta^2 - 4\alpha\gamma = 0$, both roots are the same.

Now let's verify an important fact. Let the matrix A be diagonal so that $a_{12} = a_{21} = 0$. Then the eigenvalues are

$$\begin{aligned} \lambda_1, \lambda_2 &= \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4a_{11}a_{22}}}{2} \\ &= \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} - a_{22})^2}}{2} \\ &= a_{11}, a_{22} \end{aligned}$$

For a diagonal matrix the eigenvalues are equal to the diagonal entries, in this case $\lambda_1 = a_{11}$ and $\lambda_2 = a_{22}$. Hence the stability properties of a diagonal system are directly linked to the magnitudes of the eigenvalues (whether each $|\lambda|$ is bigger or less than one).

Some more definitions: an equivalent way of saying that a matrix B is **non-singular** is to say that the only solution of the equation $Bx = 0$ is $x = 0$. Equivalently, B is singular if and only if $Bx = 0$ has solutions other than $x = 0$. Hence if λ is an eigenvalue of A such that $B = A - \lambda I$ is singular, there must be solutions other than $x = 0$ to the equation $Bx = (A - \lambda I)x = 0$. Equivalently, if λ is an eigenvalue of A, there must be at least one vector $x \neq 0$, called an **eigenvector**, such that

$$Ax = \lambda x, \qquad x \neq 0$$

Now let's return to the matter of finding diagonal matrices. (I promise that eigenvectors are involved, somehow). We want to write a matrix A in the form $AQ = QD$ for some invertible matrix Q. Supposing that A is 2-by-2, we can write this as

$$A \begin{pmatrix} q_1 & q_2 \end{pmatrix} = \begin{pmatrix} q_1 & q_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

for two vectors q_1 and q_2. From our discussion above, it's clear that the diagonal entries of D are its eigenvalues. We can write this problem one vector at a time

$$\begin{aligned} Aq_1 &= \lambda_1 q_1 \\ Aq_2 &= \lambda_2 q_2 \end{aligned}$$

So q_1 and q_2 are the eigencectors corresponding to the eigenvalues λ_1 and λ_2 of the matrix A. Then so long as the eigenvalues are distinct, $\lambda_1 \neq \lambda_2$, we can stack the eigenvectors as the columns of Q and have a diagonal matrix D with the eigenvalues of A on the diagonal! (If the eigenvalues repeat, we can't quite do this — but we won't worry about this potential difficulty here).

B. Executive summary

To sum up, if the matrix A is well-behaved (having distinct eigenvalues) we can diagonalize it by $A = QDQ^{-1}$ where D is a diagonal matrix with entries equal to the eigenvalues of A and Q is an invertible matrix whose columns are the eigenvectors corresponding to the values in D. Then if we have the system of difference equations

$$x_{t+1} - Ax_t = b, \qquad t \geq 0$$

we have the following procedure:

Step 1. Compute the steady state

$$\bar{x} = (I - A)^{-1} b$$

Step 2. Set up the equivalent homogeneous system by defining

$$y_t \equiv x_t - \bar{x}$$

so that

$$\begin{aligned} y_{t+1} = x_{t+1} - \bar{x} &= Ax_t + b - \bar{x} \\ &= Ay_t + A\bar{x} + b - \bar{x} \\ &= Ay_t + (A - I)(I - A)^{-1}b + b \\ &= Ay_t - b + b \\ &= Ay_t \end{aligned}$$

Step 3. If A has all distinct eigenvalues, use the change of variables $y_t \equiv Qz_t$ to find

$$Qz_{t+1} = AQz_t$$

or

$$z_{t+1} = Q^{-1}AQz_t = Dz_t$$

The uncoupled system has solution

$$z_t = D^t z_0$$

Step 4. Write this in the coupled variables $z_t = Q^{-1}y_t$, or

$$y_t = QD^tQ^{-1}y_0$$

Step 5. Finally, write this in terms of the original variables $x_t = y_t + \bar{x}$ or

$$\begin{aligned} x_t &= QD^tQ^{-1}(x_0 - \bar{x}) + \bar{x} \\ &= (I - QD^tQ^{-1})\bar{x} + QD^tQ^{-1}x_0 \end{aligned}$$

The stability properties are now easy to determine. If all of the eigenvalues of A are less than one in absolute value, the matrix $D^t \to 0$ as $t \to \infty$ so that $x_t \to \bar{x}$. In this case, the system is (globally) stable. Otherwise, if **any** eigenvalues of A are larger than one in absolute value, the system is unstable.

Finally, it's instructive to write the solution of the homogeneous equation as

$$y_t = Qz_t = QD^t z_0$$

or

$$y_t = \sum_{i=1}^{n} q_i \lambda_i^t z_{i,0}$$

That is the solution y_t is a weighted average of the eigenvalues λ_i with weights corresponding to the eigenvectors q_i times the corresponding initial condition $z_{i,0}$. Here we see explicitly that if any λ_i is bigger than one in absolute value, the vector y_t will diverge unless the "weight" given to that eigenvalue is zero.

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