316-406 ADVANCED MACROECONOMIC TECHNIQUES NOTE 2a

Chris Edmond ⟨cpedmond@unimelb.edu.au⟩

Optimal growth model

The key difference between the Solow growth model and the optimal or Ramsey-Cass-Koopmans growth model is that savings behavior is endogenized. We posit a single representative consumer that has preferences over an infinite stream of consumption $c = \{c_t\}_{t=0}^{\infty}$ given by a **time-separable** utility function of the form

$$u(c) = \sum_{t=0}^{\infty} \beta^t U(c_t)$$

The number β is known as the time discount factor and is usually assumed to be $0 < \beta < 1$. The period utility function $U(c)$ is assumed to be strictly increasing and concave. An important implication of time-separability is that the the marginal utility of consumption at date t

$$\frac{\partial u(c)}{\partial c_t} = \beta^t U'(c_t)$$

is independent of the level of consumption at any other date.

Abstracting from population growth or technological progress (these are easy to reinstate), the **resource constraints** facing the representative consumer are for each t

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \qquad k_0 \text{ given}$$

If output is not consumed, it is invested. Of course we maintain the usual constant returns and concavity assumptions for the intensive production function f. The rate of physical depreciation is constant at $0 < \delta < 1$.

A. Optimization

Later in this course, we will examine powerful methods for solving dynamic optimization problems. For now, let's derive first order conditions using the method of Lagrange multipliers. For each date t, let $\mu_t \geq 0$ denote the multiplier on the resource constraint. Then the Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t) + \sum_{t=0}^{\infty} \mu_t [f(k_t) + (1-\delta)k_t - c_t - k_{t+1}]$$

The first order conditions for this problem include

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Longleftrightarrow \beta^t U'(c_t) = \mu_t$$

and

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Longleftrightarrow \mu_t = \mu_{t+1}[f'(k_{t+1}) + 1 - \delta]$$

plus the resource constraints themselves. We also need the "**transversality condition**"

$$\lim_{t\to\infty} \mu_t k_{t+1} = \lim_{t\to\infty} \beta^t U'(c_t) k_{t+1} = 0$$

which requires that asymptotically the shadow value of more capital is zero. This is the natural infinite-horizon equivalent of the requirement that $k_{T+1} = 0$ in a model with a finite horizon T. Putting together the first order conditions, we have

$$\frac{\beta U'(c_{t+1})}{U'(c_t)} = \frac{\mu_{t+1}}{\mu_t} = \frac{1}{1 + f'(k_{t+1}) - \delta}$$

which requires the equality of the marginal rate of substitution between consumption today and tomorrow with the physical marginal rate of transformation. This optimality condition and ones like it are often known as **consumption Euler equations**.

An equilibrium is consumption c_t and capital k_t that solve the coupled system of non-linear difference equations

$$U'(c_t) = \beta U'(c_{t+1})[1 + f'(k_{t+1}) - \delta] \quad (1)$$

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad (2)$$

with two boundary conditions, the given initial condition k_0 and the transversality condition.

B. Steady state

The steady state is given by numbers $\bar{c} = c_t = c_{t+1}$ and $\bar{k} = k_t = k_{t+1}$. Evidently, these solve

$$1 = \beta[1 + f'(\bar{k}) - \delta]$$

$$\bar{c} + \bar{k} = f(\bar{k}) + (1-\delta)\bar{k}$$

Notice from the first equation that this means we can solve for the steady state capital stock independently of consumption (this is an artifact of the time-seperable preferences). Specifically,

$$f'\left(\bar{k}\right)=\frac{1}{\beta}-1+\delta=\rho+\delta$$

where the parameter ρ such that $\beta\equiv\frac{1}{1+\rho}$ is known as the **time discount rate**. Hence at steady state, the capital stock is such that the net marginal product of capital is equal to the discount rate. For example, if the production function is Cobb-Douglas, $f(k)=Ak^{\alpha}$, then

$$\bar{k}=\left(\frac{\alpha A}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}$$

Clearly, more patience (lower ρ) tends to increase capital accumulation and so increase $\bar{k}$. Similarly, lower depreciation δ or more capital intensity in production (higher α) raise $\bar{k}$.

Once the steady state capital stock is computed, the associated consumption level can be backed out from the resource constraint

$$\bar{c}=f\left(\bar{k}\right)-\delta\bar{k}$$

The term $\delta\bar{k}$ corresponds to steady state investment.

C. Qualitative dynamics

The consumption Euler equation says that consumption will be growing along an optimal path whenever

$$\begin{aligned}
c_{t+1}>c_t \quad &\Longleftrightarrow \quad \frac{U'(c_t)}{U'(c_{t+1})}>1\\
&\Longleftrightarrow \quad \beta[1+f'\left(k_{t+1}\right)-\delta]>1\\
&\Longleftrightarrow \quad f'\left(k_{t+1}\right)>\frac{1}{\beta}-1+\delta=\rho+\delta=f'\left(\bar{k}\right)\\
&\Longleftrightarrow \quad k_{t+1}<\bar{k}
\end{aligned}$$

Whenever the capital stock will be less than its steady state value, the real interest rate will be high relative to the time discount rate so the representative consumer will find it optimal to defer consumption so as to invest in capital accumulation thereby enjoying higher consumption tomorrow relative to today.

Similarly,

$$\begin{aligned} k_{t+1} > k_t &\iff f(k_t) + (1-\delta)k_t - c_t > k_t \\ &\iff f(k_t) - \delta k_t > c_t \end{aligned}$$

The capital stock grows whenever there is any output left over once consumption and depreciation have been taken out.

D. The linear approximation

We can write the dynamic system abstractly in the form

$$x_{t+1} = \Psi(x_t)$$

where $x_t = (c_t, k_t)$ and Ψ is the "**vector-valued**" function implied by the Euler equation and the resource constraint (1)-(2). Then the linear approximate system is

$$x_{t+1} = \bar{x} + \Psi'(\bar{x})(x_t - \bar{x})$$

or as a homogeneous equation in log-deviations

$$\hat{x}_{t+1} = \Psi'(\bar{x})\hat{x}_t, \qquad \hat{x}_t \equiv \frac{x_t - \bar{x}}{\bar{x}} \simeq \log\left(\frac{x_t}{\bar{x}}\right)$$

The term

$$A = \Psi'(\bar{x})$$

is a constant 2-by-2 matrix of coefficients. And after reviewing Note 2b on linear systems of difference equations, you'll become completely familiar with the idea that the stability of this linear system depends on the magnitudes of the eigenvalues of $\Psi'(\bar{x})$.

Before going any further, it's worth explicitly log-linearizing the system of equations to investigate the properties of the coefficient matrix. The log-linear equations can be written (and you should definitely verify these calculations)

$$\hat{c}_{t+1} - \frac{\beta f''(\bar{k})\bar{k}}{\mathcal{R}(\bar{c})}\hat{k}_{t+1} = \hat{c}_t$$

and

$$\hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t - \frac{\bar{c}}{\bar{k}}\hat{c}_t$$

where the number

$$\mathcal{R}(c) \equiv -\frac{U''(c)c}{U'(c)} \geq 0$$

is the so-called Arrow/Pratt measure of relative risk aversion (i.e., a measure of the local concavity of the utility function).

In Matrix form, these two equations are

$$\begin{pmatrix} 1 & -\frac{\beta f''(\bar{k})\bar{k}}{\mathcal{R}(\bar{c})} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\bar{c}}{\bar{k}} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

and on rearranging[1]

$$\begin{aligned} \begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} &= \begin{pmatrix} 1 & \frac{\beta f''(\bar{k})\bar{k}}{\mathcal{R}(\bar{c})} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\bar{c}}{\bar{k}} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{\beta f''(\bar{k})\bar{c}}{\mathcal{R}(\bar{c})} & -\frac{\beta f''(\bar{k})\bar{k}}{\mathcal{R}(\bar{c})}\frac{1}{\beta} \\ -\frac{\bar{c}}{\bar{k}} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix} \end{aligned}$$

To determine the stability properties of the model, we need to know the eigenvalues (λ_1, λ_2) of the 2-by-2 coefficient matrix A, where

$$A \equiv \begin{pmatrix} 1 - \frac{\beta f''(\bar{k})\bar{c}}{\mathcal{R}(\bar{c})} & -\frac{\beta f''(\bar{k})\bar{k}}{\mathcal{R}(\bar{c})}\frac{1}{\beta} \\ -\frac{\bar{c}}{\bar{k}} & \frac{1}{\beta} \end{pmatrix}$$

Two important results in linear algebra are that the trace[2] and determinant are respectively the

[1] Here I use the following result for 2-by-2 matrices (this does not generalize to higher dimensional square matrices). Suppose that C is invertible with

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

Then

$$C^{-1} = \frac{1}{\det(C)} \begin{pmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{pmatrix}$$

where $\det(C) = c_{11}c_{22} - c_{12}c_{21}$.

[2] The **trace** of a matrix is the sum of its diagonal elements.

sum and product of the eigenvalues (λ_1, λ_2). So

$$\begin{aligned}
\text{tr}(A) &= \lambda_1 + \lambda_2 = 1 - \frac{\beta f''(\bar{k})\bar{c}}{\mathcal{R}(\bar{c})} + \frac{1}{\beta} > 2 \\
\det(A) &= \lambda_1 \times \lambda_2 = \frac{1}{\beta} > 1
\end{aligned}$$

The discriminant of A is positive, specifically

$$\begin{aligned}
\Delta &\equiv \text{tr}(A)^2 - 4\det(A) \\
&= \left(1 - \frac{\beta f''(\bar{k})\bar{c}}{\mathcal{R}(\bar{c})} + \det(A)\right)^2 - 4\det(A) \\
&= \left(1 - \frac{\beta f''(\bar{k})\bar{c}}{\mathcal{R}(\bar{c})} - \det(A)\right)^2 > 0
\end{aligned}$$

so the eigenvalues are both real. Also, because the product of eigenvalues $\det(A)$ is positive, they must both have the same sign. Since the sum of the eigenvalues $\text{tr}(A)$ is also positive, and they are both of the same sign, both eigenvalues must individually be positive.

We can also establish the magnitudes of the eigenvalues in the following manner. The eigenvalues are the roots of the characteristic polynomial of A

$$p(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

But

$$p(1) = (1 - \lambda_1)(1 - \lambda_2) = 1 - \text{tr}(A) + \det(A) = -\frac{\beta f''(\bar{k})\bar{c}}{\mathcal{R}(\bar{c})} < 0$$

So both eigenvalues are positive but $(1 - \lambda_1)(1 - \lambda_2) < 0$. This can only be true if one eigenvalue is less than one and the other eigenvalue is greater than one, say $\lambda_1 > 1$ and $0 < \lambda_2 < 1$. We have formally established what we already guessed from the phase diagram: the linear system is **saddle-path unstable**.

E. Solving for the transitional dynamics

To summarize, we have a dynamic system that can be written in log-deviations as

$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} = A \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

where A is a 2-by-2 matrix of coefficients with one unstable eigenvalue $\lambda_1 > 1$ and one stable eigenvalue $0 < \lambda_2 < 1$. The solution to a system like this can be written

$$\begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix} = Q \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} Q^{-1} \begin{pmatrix} \hat{c}_0 \\ \hat{k}_0 \end{pmatrix}$$

where Q is the matrix of eigenvectors with columns corresponding to the eigenvalues and the initial condition $\hat{k}_0$ is given. We have one explosive eigenvalue and the matrix Q depends only on given parameters of the model, so this system will blow-up as $t \to \infty$ ... unless ...

...unless the endogenous variable $\hat{c}_0$ "**jumps**" in just the right way so as to kill the explosive dynamics. To see how this works write the system out as

$$\begin{aligned}
\begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix} &= \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} \frac{1}{\det(Q)} \begin{pmatrix} q_{22} & -q_{12} \\ -q_{21} & q_{11} \end{pmatrix} \begin{pmatrix} \hat{c}_0 \\ \hat{k}_0 \end{pmatrix} \\
&= \begin{pmatrix} q_{11}\lambda_1^t & q_{12}\lambda_2^t \\ q_{21}\lambda_1^t & q_{22}\lambda_2^t \end{pmatrix} \begin{pmatrix} q_{22}\hat{c}_0 - q_{12}\hat{k}_0 \\ -q_{21}\hat{c}_0 + q_{11}\hat{k}_0 \end{pmatrix} \frac{1}{\det(Q)} \\
&= \begin{pmatrix} q_{11}\lambda_1^t(q_{22}\hat{c}_0 - q_{12}\hat{k}_0) - q_{12}\lambda_2^t(q_{21}\hat{c}_0 - q_{11}\hat{k}_0) \\ q_{21}\lambda_1^t(q_{22}\hat{c}_0 - q_{12}\hat{k}_0) - q_{22}\lambda_2^t(q_{21}\hat{c}_0 - q_{11}\hat{k}_0) \end{pmatrix} \frac{1}{\det(Q)}
\end{aligned}$$

Since $\lambda_1^t \to \infty$ as $t \to \infty$, we have to shut these explosive paths down by setting $\hat{c}_0$ just right (this is our one degree of freedom, it is the only endogenous variable not otherwise pinned down). Essentially, we're picking consumption so that we do not have an explosive path that violates either implicit non-negativity constraints on consumption and capital or the transversality condition. Apparently, setting

$$\hat{c}_0 = \frac{q_{12}}{q_{22}} \hat{k}_0$$

will ensure $q_{22}\hat{c}_0 - q_{12}\hat{k}_0 = 0$ and wipe out the unstable dynamics.

Hence our complete solution is

$$\begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix} = \frac{1}{\det(Q)} \begin{pmatrix} q_{12}(q_{11} - q_{21}\frac{q_{12}}{q_{22}}) \\ q_{22}(q_{11} - q_{21}\frac{q_{12}}{q_{22}}) \end{pmatrix} \lambda_2^t \hat{k}_0$$

In practice this means we can first compute the matrix A, then compute its eigenvalues and eigenvectors to get the matrix Q and pick the stable eigenvalue so that we have a bounded solution.

F. Method of undetermined coefficients

Notice that both solutions can be written as linear difference equations

$$\begin{aligned}\hat{c}_{t+1} &= \lambda\hat{c}_t \\ \hat{k}_{t+1} &= \lambda\hat{k}_t\end{aligned}$$

for some common as yet unknown coefficient λ. This inspires a procedure which doesn't involve the hum-drum of the matrix calculations (though it makes use of facts that we only know are true only because we bothered to do the hard work).

Guess that the solution takes the form of linear difference equations with unknown coefficient. Then take the system of log-linear equations as before

$$\begin{aligned}\hat{c}_{t+1} - \frac{\beta f''(\bar{k})\bar{k}}{\mathcal{R}(\bar{c})}\hat{k}_{t+1} - \hat{c}_t &= 0 \\ \hat{k}_{t+1} - \frac{1}{\beta}\hat{k}_t + \frac{\bar{c}}{\bar{k}}\hat{c}_t &= 0\end{aligned}$$

and plug in the hypothetical solutions to get

$$\begin{aligned}\lambda\hat{c}_t - \frac{\beta f''(\bar{k})\bar{k}}{\mathcal{R}(\bar{c})}\lambda\hat{k}_t - \hat{c}_t &= 0 \\ \lambda\hat{k}_t - \frac{1}{\beta}\hat{k}_t + \frac{\bar{c}}{\bar{k}}\hat{c}_t &= 0\end{aligned}$$

Eliminating one of the variables (consumption, say) and rearranging gives

$$\left[(\lambda-1)\frac{\bar{k}}{\bar{c}}\left(\frac{1}{\beta}-\lambda\right) - \frac{\beta f''(\bar{k})\bar{k}}{\mathcal{R}(\bar{c})}\lambda\right]\hat{k}_t = 0$$

The solution λ has to hold **for every possible** $\hat{k}_t$. This can only work if the solution is such that the term in square brackets is zero. That is, λ must be such that

$$(\lambda-1)\frac{\bar{k}}{\bar{c}}\left(\frac{1}{\beta}-\lambda\right) - \frac{\beta f''(\bar{k})\bar{k}}{\mathcal{R}(\bar{c})}\lambda = 0$$

But on rearranging this, we have the same characteristic polynomial as before

$$p(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

where

$$
\begin{aligned}
\text{tr}(A) &= 1 - \frac{\beta f''(\bar{k})\bar{c}}{\mathcal{R}(\bar{c})} + \frac{1}{\beta} \\
\det(A) &= \frac{1}{\beta}
\end{aligned}
$$

In short, we can find the unique bounded solution to the log-linear optimal growth model by solving this quadratic equation and choosing the stable root.

An example with a closed form solution

One particular parametric example of the optimal growth model can be solved in closed form (i.e., without any need for approximations). Let the period utility function be $U(c) = \log(c)$ and let the production function be $f(k) = Ak^{\alpha}$ with full depreciation, $\delta = 1$. Then the optimal consumption and capital paths are given by

$$
\begin{aligned}
c_t &= (1-\alpha\beta)Ak_t^{\alpha} \\
k_{t+1} &= \alpha\beta Ak_t^{\alpha}, \qquad k_0 \text{ given}
\end{aligned}
$$

Clearly these satisfy the resource constraint

$$
c_t + k_{t+1} = (1-\alpha\beta)Ak_t^{\alpha} + \alpha\beta Ak_t^{\alpha} = Ak_t^{\alpha} = f(k_t)
$$

They also satisfy the transversality condition

$$
\begin{aligned}
\lim_{t\to\infty} \beta^t U'(c_t)k_{t+1} &= \lim_{t\to\infty} \beta^t \frac{1}{(1-\alpha\beta)Ak_t^{\alpha}} k_{t+1} \\
&= \lim_{t\to\infty} \beta^t \frac{\alpha\beta Ak_t^{\alpha}}{(1-\alpha\beta)Ak_t^{\alpha}} \\
&= \lim_{t\to\infty} \beta^t \frac{\alpha\beta}{1-\alpha\beta} \\
&= 0
\end{aligned}
$$

since $0 < \beta < 1$ so that $\beta^t \to 0$. All that's left is to check the Euler equation, which can be written

$$
\begin{aligned}
\frac{U'(c_t)}{U'(c_{t+1})} = \beta f'(k_{t+1}) &\iff \frac{c_{t+1}}{c_t} = \alpha\beta Ak_{t+1}^{\alpha-1} \\
&\iff \frac{(1-\alpha\beta)Ak_{t+1}^{\alpha}}{(1-\alpha\beta)Ak_t^{\alpha}} = \alpha\beta Ak_{t+1}^{\alpha-1}
\end{aligned}
$$

$$\begin{aligned} \iff & \quad \frac{1}{k_t^\alpha} = \alpha\beta A \frac{1}{k_{t+1}} \\ \iff & \quad k_{t+1} = \alpha\beta A k_t^\alpha \end{aligned}$$

as required. So we have indeed a solution (and it is the only one). In this example, the consumption function

$$c_t = (1 - \alpha\beta) A k_t^\alpha$$

gives an explicit formula for the **stable arm** of the saddle path. In other contexts, this is sometimes known as the "**policy function**". With this formula, we know the unique $c_0 = (1 - \alpha\beta) A k_0^\alpha$ corresponding to the initial capital stock so that economy jumps onto the saddle path.

Chris Edmond

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