316-406 ADVANCED MACROECONOMIC TECHNIQUES NOTE 1c

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Linearizing a difference equation

We will frequently want to linearize a difference equation of the form

$$x_{t+1} = \psi(x_t)$$

given an initial condition x_0.

We can take a first order approximation to the function ψ at some candidate point $\tilde{x}$. This gives

$$x_{t+1} \simeq \psi(\tilde{x}) + \psi'(\tilde{x})(x_t - \tilde{x})$$

The right hand side of this expression is a linear equation in x_t with slope $\psi'(\tilde{x})$ equal to the derivative of ψ evaluated at the point $\tilde{x}$. Typically, we will want to linearize around a steady state $\bar{x}$ which also has the property $\bar{x} = \psi(\bar{x})$ so that

$$x_{t+1} \simeq \bar{x} + \psi'(\bar{x})(x_t - \bar{x})$$

If we treat this approximation as exact, we have an inhomogenous difference equation with constant coefficients of the form

$$x_{t+1} = ax_t + b$$

where

$$\begin{aligned} a &\equiv \psi'(\bar{x}) \\ b &\equiv (1-a)\bar{x} \end{aligned}$$

And we know that the solution to this difference equation is

$$x_t = (1-a^t)\bar{x} + a^t x_0$$

Hence if $|a| < 1$, then $a^t \to 0$ as $t \to \infty$ so that $x_t \to \bar{x}$. We can conclude from this that the original non-linear difference equation is **locally stable** (since we have had to take a local approximation to the function ψ) but we cannot use this method to determine global properties of the solution —

in a linear model, local and global stability are the same thing, but that is not true here. A lot of applied work in economics works by using linear approximations to non-linear models.

Log-linearization, etc.

A popular alternative to linearizing a model is to log-linearize it. To do this, define the log-deviation of a variable from its steady state value as

$$\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right)$$

With this notation, a variable is at steady state when its log-deviation is zero.

The popularity of this method comes from the units-free nature of the variables. Log-deviations are approximate percentage deviations from steady state and the coefficients of log-linear models are elasticities. I say approximate percentage deviation because

$$\log\left(\frac{x_t}{\bar{x}}\right) = \log\left(1 + \frac{x_t - \bar{x}}{\bar{x}}\right) \simeq \frac{x_t - \bar{x}}{\bar{x}}$$

This uses the first order approximation

$$\log(1+z) \simeq \log(1+z_0) + \frac{1}{1+z_0}(z - z_0)$$

around the point $z_0 = 0$, so it is an approximation that is valid when the deviation is small. See "Problem Set Zero" for some discussion of the accuracy of this approximation.

A non-linear difference equation of the form

$$x_{t+1} = \psi(x_t)$$

is log-linearized by making the change of variables $x_t = \bar{x}\exp(\hat{x}_t)$ and then linearizing both sides with respect to $\hat{x}_t$ around the point $\hat{x}_t = 0$. Specifically, write

$$\bar{x}\exp(\hat{x}_{t+1}) = \psi(\bar{x}\exp(\hat{x}_t))$$

and now linearize both sides of this equality with respect to $\hat{x}_t$ (around the point $\hat{x}_t = 0$) to get

$$\bar{x}\exp(0) + \bar{x}\exp(0)(\hat{x}_{t+1} - 0) \simeq \psi(\bar{x}\exp(0)) + \psi'(\bar{x}\exp(0))\bar{x}\exp(0)(\hat{x}_t - 0)$$

Treating this approximation as exact and simplifying gives

$$\bar{x} + \bar{x}\hat{x}_{t+1} = \psi(\bar{x}) + \psi'(\bar{x})\bar{x}\hat{x}_t$$

And recognizing that $\bar{x} = \psi(\bar{x})$ gives

$$\hat{x}_{t+1} = \psi'(\bar{x})\hat{x}_t$$

Local stability again crucially depends on the absolute magnitude of $\psi'(\bar{x})$, with $|\psi'(\bar{x})| < 1$ giving a locally stable steady state.

Suppose, for example, that we have the Solow difference equation

$$k_{t+1} = \psi(k_t) \equiv \frac{s}{(1+g)(1+n)} k_t^{\alpha} + \frac{(1-\delta)}{(1+g)(1+n)} k_t$$

Then the log-linear approximation is

$$\hat{k}_{t+1} = \psi'(\bar{k})\hat{k}_t = \frac{1}{(1+g)(1+n)}[s\alpha\bar{k}^{\alpha-1} + 1 - \delta]\hat{k}_t$$

Calculus for log-linearizations

Similar derivations allow us to do more complicated log-linearizations (this is sometimes known as the "Campbell calculus"). The following basic rule often helps immensely. Suppose that we have a differentiable function

$$y_t = f(x_t, z_t)$$

(more arguments will generalize in an obvious fashion). Then the log-linear approximation is

$$\bar{y}\hat{y}_t = f_x(\bar{x}, \bar{z})\bar{x}\hat{x}_t + f_z(\bar{x}, \bar{z})\bar{z}\hat{z}_t$$

which is often written

$$\hat{y}_t = \frac{f_x(\bar{x}, \bar{z})\bar{x}}{f(\bar{x}, \bar{z})}\hat{x}_t + \frac{f_z(\bar{x}, \bar{z})\bar{z}}{f(\bar{x}, \bar{z})}\hat{z}_t$$

so that the coefficients on the log-deviations $\hat{x}_t$, $\hat{z}_t$ are elasticities. A 1% increase in $\hat{x}_t$ near the steady-state gives approximately a $\frac{f_x(\bar{x},\bar{z})\bar{x}}{f(\bar{x},\bar{z})}100\%$ increase in $\hat{y}_t$. Here are some further applications of this rule.

1. (Multiplication and division). Let

$$y_t = f(x_t, z_t) = x_t z_t$$

Then

$$\begin{aligned} \hat{y}_t = \frac{f_x(\bar{x}, \bar{z})\bar{x}}{f(\bar{x}, \bar{z})}\hat{x}_t + \frac{f_z(\bar{x}, \bar{z})\bar{z}}{f(\bar{x}, \bar{z})}\hat{z}_t &= \frac{\bar{z}\bar{x}}{\bar{x}\bar{z}}\hat{x}_t + \frac{\bar{x}\bar{z}}{\bar{x}\bar{z}}\hat{z}_t \\ &= \hat{x}_t + \hat{z}_t \end{aligned}$$

And similarly, if

$$y_t = f(x_t, z_t) = \frac{x_t}{z_t}$$

Then

$$\hat{y}_t = \hat{x}_t - \hat{z}_t$$

2. (Addition and subtraction). Let

$$y_t = f(x_t, z_t) = x_t + z_t$$

Then

$$\begin{aligned} \bar{y}\hat{y}_t &= f_x(\bar{x}, \bar{z})\bar{x}\hat{x}_t + f_z(\bar{x}, \bar{z})\bar{z}\hat{z}_t = 1\bar{x}\hat{x}_t + 1\bar{z}\hat{z}_t \\ &= \bar{x}\hat{x}_t + \bar{z}\hat{z}_t \end{aligned}$$

3. (Implicit functions). An important special case concerns functions of the form

$$0 = g(x_t, y_t)$$

which **may implicitly** defines y_t in terms of x_t (or vice-versa). If $g_y(\bar{x}, \bar{y}) \neq 0$, then

$$0 = g_x(\bar{x}, \bar{y})\bar{x}\hat{x}_t + g_y(\bar{x}, \bar{y})\bar{y}\hat{y}_t$$

and rearranging gives

$$\hat{y}_t = -\frac{g_x(\bar{x}, \bar{y})\bar{x}}{g_y(\bar{x}, \bar{y})\bar{y}}\hat{x}_t$$

Examples

1. (Difference equation). Let $x_{t+1} = f(x_t)$ so that

$$\bar{x}\hat{x}_{t+1} = f'(\bar{x})\bar{x}\hat{x}_t$$

The steady state values on both sides cancel and we are left with

$$\hat{x}_{t+1} = f'(\bar{x})\hat{x}_t$$

2. (Constant elasticity function). Let $z = x^{\varepsilon}$ for some coefficient ε. Then

$$\begin{aligned} \bar{z}\hat{z}_t &= \varepsilon\bar{x}^{\varepsilon-1}\bar{x}\hat{x}_t \\ &= \varepsilon\bar{x}^{\varepsilon}\hat{x}_t \\ \implies & \hat{z}_t = \varepsilon\frac{\overline{x}^{\varepsilon}}{\overline{z}}\hat{x}_t = \varepsilon\hat{x}_t \end{aligned}$$

In this case, the elasticity does not depend on the steady state values $\bar{x}, \bar{z}$ but only on the constant parameter ε.

3. (Consumption Euler equation — an extended example). A consumer's first order condition is often

$$1 = \beta\frac{U'(c_{t+1})}{U'(c_t)}R_{t+1}$$

This implies the log-linearization

$$\hat{1} = \hat{\beta} + \widehat{U'(c_{t+1})} - \widehat{U'(c_t)} + \hat{R}_{t+1}$$

But since the log-deviations of constants are zero, this is just

$$0 = \widehat{U'(c_{t+1})} - \widehat{U'(c_t)} + \hat{R}_{t+1}$$

Moreover, if we have the constant elasticity utility function $U(c) = c^{1-\sigma}$ with coefficient $\sigma > 0$, then marginal utility is $U'(c) = c^{-\sigma}$ and the log-deviation of marginal utility is

$$\widehat{U'(c_t)} = \frac{U''(\bar{c})\bar{c}}{U'(\bar{c})}\hat{c}_t = -\sigma\hat{c}_t$$

In this example, the consumption Euler equation becomes

$$\hat{c}_{t+1} - \hat{c}_t = \frac{1}{\sigma}\hat{R}_{t+1}$$

The coefficient $\frac{1}{\sigma} > 0$ measures the sensitivity of consumption growth to real interest rates in excess of their steady state value. Also, if we write the gross real interest rate R_{t+1} as one plus the net rate, $R_{t+1} = 1 + r_{t+1}$, then

$$\bar{R}\hat{R}_{t+1} = \bar{r}\hat{r}_{t+1}$$

But in steady state $c_t = c_{t+1} = \bar{c}$ and so $\bar{R} = \beta^{-1} = 1 + \bar{r}$ (according to the original Euler equation). Hence

$$\hat{R}_{t+1} = (1-\beta)\hat{r}_{t+1}$$

4. (National income accounting). Let

$$c_t + i_t = y_t$$

This implies the log-linearization

$$\bar{c}\hat{c}_t + \bar{\imath}\hat{\imath}_t = \bar{y}\hat{y}_t$$

which is often written so that the coefficients are shares of GDP, that is

$$\frac{\bar{c}}{\bar{y}}\hat{c}_t + \frac{\bar{\imath}}{\bar{y}}\hat{\imath}_t = \hat{y}_t$$

5. (Cobb-Douglas production function). Let

$$y_t = f(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha}, \qquad 0 < \alpha < 1$$

Then

$$\hat{y}_t = \alpha\hat{k}_t + (1-\alpha)\hat{n}_t$$

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