316-406 ADVANCED MACROECONOMIC TECHNIQUES NOTE 1b

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Linear differential equations with constant coefficients

We will frequently want to solve a differential equation of the form

$$\dot{x}_t - ax_t = b, \qquad t \geq 0$$

given scalars a, b and an initial condition x_0.

A. Homogenous equations

If $b = 0$, we have the homogenous equation

$$\dot{x}_t - ax_t = 0, \qquad t \geq 0$$

Introspection tells us that the solution is a function x_t that grows or decays exponentially at rate a (growing if $a > 0$, decaying if $a < 0$). That is, we expect the solution to be

$$x_t = e^{at}x_0, \qquad t \geq 0$$

And differentiating this with respect to time verifies that it is indeed the solution (there is only one). To see why this is the solution, write the problem as

$$\frac{dx_s}{x_s} = ads, \qquad s \geq 0$$

and integrate both sides over the interval $[0, t)$. This gives

$$\log(x_t) - \log(x_0) = \int_0^t \frac{dx_s}{x_s} = \int_0^t ads = at$$

And rearranging gives

$$x_t = e^{at}x_0, \qquad t \geq 0$$

This solution diverges to $\pm\infty$ (depending on the sign of x_0) if $a > 0$ but instead decays to zero if $a < 0$. Put differently, the differential equation is **unstable** if $a > 0$ but **stable** if $a < 0$.

B. Inhomogenous equations

Otherwise, if $b \neq 0$, we have to do a bit more work. The trick is to transform the inhomogenous equation into a homogenous equation by a change of variables. Denote by $\bar{x}$ that unique value of x_t such that $\dot{x}_t = 0$. This is the **steady state** of x_t. Clearly,

$$\bar{x} = -\frac{b}{a}$$

which is well defined so long as $a \neq 0$. Now introduce a new variable

$$y_t \equiv x_t - \bar{x}$$

(i.e., the difference between the actual and steady state value of x_t). Note that this change of variables implies

$$\begin{aligned}
\dot{y}_t = \dot{x}_t &= ax_t + b \\
&= a(y_t + \bar{x}) + b \\
&= ay_t - b + b \\
&= ay_t
\end{aligned}$$

So the new variable obeys a homogenous differential equation and therefore has the solution

$$y_t = e^{at} y_0, \qquad t \geq 0$$

Plugging back in the definition $y_t \equiv x_t - \bar{x}$ gives

$$(x_t - \bar{x}) = e^{at}(x_0 - \bar{x}), \qquad t \geq 0$$

which is sometimes re-written

$$x_t = (1 - e^{at})\bar{x} + e^{at} x_0, \qquad t \geq 0$$

C. Stability

The stability properties of this solution are easy. If $a < 0$, then $e^{at} \to 0$ as $t \to \infty$ so that $x_t \to \bar{x}$ irrespective of the value of the initial condition. That is, if $a < 0$, the steady state $\bar{x}$ is **globally**

stable. If $a > 0$, then $e^{at} \to +\infty$ as $t \to \infty$ so that $x_t \to \pm\infty$ depending on the sign of $x_0 - \bar{x}$, i.e., depending on whether the variable starts above or below its steady state value. If the initial condition happens to be $x_0 = \bar{x}$, the system stays there irrespective of the value of a. Finally, if in fact $a = 0$, then we have the trivial differential equation

$$\dot{x}_t = b, \qquad t \geq 0$$

with general solution

$$x_t = tb + x_0, \qquad t \geq 0$$

Linear difference equations

Similarly, if we have the linear difference equation

$$x_{t+1} - ax_t = b, \qquad t = 0, 1, 2, ...$$

given scalars a, b and an initial condition x_0, then the homogenous solution (when $b = 0$) is

$$x_t = a^t x_0, \qquad t = 0, 1, 2, ...$$

You can verify this by iterating as follows

$$\begin{aligned} x_1 &= a^1 x_0 \\ x_2 &= a^1 x_1 = a^2 x_0 \\ &\vdots \\ x_t &= a^1 x_{t-1} = a^t x_0 \end{aligned}$$

The general solution when $b \neq 0$ is

$$(x_t - \bar{x}) = a^t (x_0 - \bar{x}), \qquad t = 0, 1, 2, ...$$

or

$$x_t = (1 - a^t)\bar{x} + a^t x_0, \qquad t = 0, 1, 2, ...$$

But in the discrete time model, the steady state $\bar{x}$ is the solution to $\bar{x} - a\bar{x} = b$ or

$$\bar{x} = \frac{b}{1-a}$$

which is well defined so long as $a \neq 1$. If $a = 1$, there is no steady state but we have the solution

$$x_t = tb + x_0, \qquad t = 0, 1, 2, ...$$

In discrete time, stability properties are determined by whether $|a| < 1$ or not. If $|a| < 1$, then $a^t \to 0$ as $t \to \infty$ so that $x_t \to \bar{x}$ irrespective of the value of the initial condition. If $|a| > 1$, then $a^t \to \pm\infty$ as $t \to \infty$, so the steady state is not stable. Notice that if $a > 0$, the motion of x_t is monotonic but if $a < 0$, the motion of x_t is oscillatory. Again, if the initial condition happens to be $x_0 = \bar{x}$, the system stays there irrespective of the value of a.

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