## 316-406/671 ADVANCED MACRO TECHNIQUES

This exam lasts **180 minutes** and has two questions. The first question is worth 120 marks, while the second question is worth only 60 marks. Allocate your time accordingly. Within each question there are a number of parts and the weight given to each part is also indicated. Even if you cannot complete one part of a question, you should be able to move on an answer other parts, so do not spend too much time. If you feel like you are getting stuck, move on to the next part. You also have **15 minutes perusal** before you can start writing answers.

**Question 1.** *Term Structure of Interest Rates* (120 marks): Consider a representative agent asset pricing model where preferences are

$$\mathsf{E}_0\left\{\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\gamma}}{1-\gamma}\right\},\qquad 0<\beta<1 \text{ and } \gamma>0$$

There are two kinds of assets. First, there is a "Lucas tree" with dividends $\{x_t\}$ that follow an autoregression of the form

$$x_{t+1}=\bar{x}^{1-\phi}x_t^{\phi}\varepsilon_{t+1},\qquad 0<\phi<1 \text{ and } \bar{x}>0 \tag{1}$$

where $\log(\varepsilon_{t+1})$ are IID normal with mean 1 and variance $\sigma^2$. Second, there are also bonds **of various maturities**. A $j$-period bond ($j\geq 1$) is a riskless claim to one unit of consumption to be delivered in $j$-period's time.

(a) (15 marks): Suppose we let bonds up-to maturity $J=2$ be traded. That is, one-period ($j=1$) and two-period ($j=2$) bonds are traded. Let $q_j(x)$ denote the price of a $j$-bond if the current aggregate state is $x$ and let $p(x)$ denote the price of a claim to the Lucas tree. Let $V(w,x)$ denote the consumer's value function if their individual wealth is $w$ and the aggregate state is $x$. Write down a Bellman equation for the consumer's problem. Be careful to explain the Bellman equation and any constraints that you provide.

(b) (15 marks): Define a **recursive competitive equilibrium** for this economy.

(c) (15 marks): Suppose we let bonds with a maturity up to an arbitrary $J$ be traded. Explain why **in equilibrium** the price of a $j$-bond satisfies the relationship

$$q_j(x_t) = \mathsf{E}_t \left\{ \beta^j \left( \frac{x_{t+j}}{x_t} \right)^{-\gamma} \right\}$$

where $\mathsf{E}_t\{\}$ denotes expectations conditional on the current aggregate state $x_t$. [*Hint*: begin by using first order and envelope conditions to derive bond prices for the $j = 1$ and $j = 2$ cases, then explain why this generalizes to arbitrary $j$].

(d) (15 marks): By iterating on equation (1), show that log-dividends in $j$ periods time satisfy

$$\log \left( \frac{x_{t+j}}{\bar{x}} \right) = \phi^j \log \left( \frac{x_t}{\bar{x}} \right) + \sum_{k=0}^{j-1} \phi^k \log(\varepsilon_{t+j-k})$$

Use this to derive an expression for dividend growth between $t$ and $t+j$, namely

$$\log \left( \frac{x_{t+j}}{x_t} \right)$$

(e) (30 marks): Now use your solutions from part (c) to solve for bond prices $q_j(x_t)$. You should make use of the following three pieces of information: First, you can write

$$q_j(x_t) = \mathsf{E}_t \left\{ \beta^j \left( \frac{x_{t+j}}{x_t} \right)^{-\gamma} \right\} = \beta^j \mathsf{E}_t \left\{ \exp \left[ -\gamma \log \left( \frac{x_{t+j}}{x_t} \right) \right] \right\}$$

Second, if $\log(\varepsilon_{t+j-k})$ is normal so is the linear combination $\sum_{k=0}^{j-1} \phi^k \log(\varepsilon_{t+j-k})$. Third, if some random variable $X$ is normal with mean $\mu_X$ and variance $\sigma_X^2$ then $\mathsf{E}\{\exp(X)\} = \exp(\mu_X + \frac{1}{2}\sigma_X^2)$.

(f) (30 marks): Define $j$-period bond **returns** by the formula $R_j(x_t) \equiv [1/q_j(x_t)]^{1/j}$. Now use your solution from part (e) to explain why $j$-period returns can be written

$$\log[R_j(x_t)] = a_j + b_j \log \left( \frac{x_t}{\bar{x}} \right)$$

for some coefficients $a_j$ and $b_j$. Provide an explicit solution for $b_j$ in terms of the parameters of the model. Explain whether $b_j$ and $|b_j|$ are increasing or decreasing in $j$. Is the variance of short term interest rates higher or lower than long term interest rates? Are short term interest rates more or less sensitive to the state of the economy (i.e., to $x_t$) than are long term interest rates? What happens to the slope of the term structure as $\phi \to 1$ (as dividends become very persistent)? Give economic intuition wherever you can.

**Question 2.** *Incomplete Markets* (60 marks). Consider a household with the problem of maximizing

$$\mathsf{E}_0\left\{\sum_{t=0}^{\infty}\beta^t U(c_t)\right\}, \qquad 0<\beta<1$$

subject to a flow budget constraint

$$c_t + a_{t+1} = (1+r)a_t + w_t$$

where $a_{t+1}$ denotes assets carried into the next period, $r$ denotes a constant interest rate, and $w_t$ is the real wage rate, which follows a Markov chain on a discrete set $\mathcal{W}$ with transition matrix $P$ and typical elements

$$P(w', w) = \Pr(w_{t+1} = w' | w_t = w)$$

Households also face the borrowing constraint

$$a_{t+1} \geq -\phi$$

(a) (10 marks): Let $V(a, w)$ denote the value function of a household with current assets $a$ facing real wage $w$. Set up a Bellman equation for the household's problem.

(b) (10 marks): Give an algorithm that explains how you would find approximate solutions by value function iteration on a discrete state space. In your answer, let $\mathcal{A} \times \mathcal{W}$ denote the discretized state space.

(c) (20 marks): Suppose that $a' = g(a, w)$ denotes the policy function that you obtain from solving your dynamic programming problem. Let $\mu_t(a, w)$ denote the unconditional distribution of $(a, w)$ pairs on $\mathcal{A} \times \mathcal{W}$. That is,

$$\mu_t(a, w) = \Pr(a_t = a, w_t = w)$$

Explain how you can use the policy function $g(a, w)$ and the transition matrix $P(w', w)$ to create a law of motion that maps $\mu_t(a, w)$ to $\mu_{t+1}(a', w')$. Give an algorithm that explains how you could solve for a stationary distribution [i.e., a time-invariant $\mu(a, w)$]. Explain why this distribution has **both** a time-series and a cross-sectional interpretation.

(d) (10 marks): Define a **stationary recursive competitive equilibrium** for this model. Your definition should involve the policy function $g$, the stationary distribution $\mu$, and the real interest rate $r$. What is the market clearing condition that characterizes $r$?

(e) (10 marks): Give an algorithm that explains how you could compute a stationary competitive equilibrium as you've defined it.

END OF EXAM