316-632 International Monetary Economics — Note 2

Consider the problem of a representative consumer in a country that is perfectly integrated with world capital markets and that has access to a riskless world real interest rate $r > 0$. The consumer is born at date $t = 0$ and lives forever and has preferences $U(c)$ over random consumption vectors

$$c = (c_0, c_1, \cdots)$$

This consumer faces a random endowment process $\{y_t\}$ and has the *flow budget constraints*

$$B_{t+1} - B_t = rB_t + y_t - c_t$$

The consumer's preferences are given by an expected utility function

$$U(c) = E_0\left\{\sum_{t=0}^{\infty} \beta^t u(c_t)\right\} \qquad 0 < \beta < 1$$

where $E_t\{w\}$ denotes the expectation of the random variable w conditional on the information set at time t. In the notation I will use, a variable subscripted t is known at date t; thus at date t, $u'(c_t)$ is known but $u'(c_{t+1})$ is not.

The Lagrangian for this problem can be written

$$L = E_0\left\{\sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda_t[(1+r)B_t + y_t - c_t - B_{t+1}]\right\}$$

where the Lagrange multipliers $\{\lambda_t\}$ on the flow constraints are generally stochastic too (because they reflect the marginal utility of wealth which depends on the stochastic process $\{y_t\}$).

The first order conditions for this problem include

$$\begin{aligned} \frac{\partial L}{\partial c_t} &= 0 \qquad \Leftrightarrow \qquad E_0 E_t\{\beta^t u'(c_t)\} = E_0 E_t\{\lambda_t\} \\ \frac{\partial L}{\partial B_{t+1}} &= 0 \qquad \Leftrightarrow \qquad E_0 E_t\{\lambda_t\} = E_0 E_t\{(1+r)\lambda_{t+1}\} \end{aligned}$$

which can be simplified using the Law of Iterated Expectations to give

$$\begin{aligned} \beta^t u'(c_t) &= \lambda_t \\ \lambda_t &= (1+r)E_t\{\lambda_{t+1}\} \end{aligned}$$

These are to be interpreted as holding for every date and state of nature (i.e., for every date and possible realization of y_t). They can be combined to give the *stochastic Euler equation*

$$u'(c_t) = \beta(1+r)E_t\{u'(c_{t+1})\}$$

which characterizes optimal consumption-smoothing behavior for this problem.

Investment

Now let's extend this problem by including production and capital accumulation. Let the production function be

$$y_t = A_t F(k_t)$$

where A_t denotes a random technology shock and k_t denotes physical capital. Capital is accumulated by making investments and does not depreciate

$$k_{t+1} = k_t + i_t$$

And the flow budget constraint of the consumer is now

$$(B_{t+1} - B_t) + (k_{t+1} - k_t) = rB_t + A_t F(k_t) - c_t$$

The consumer's Lagrangian is

$$L = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda_t [(1+r)B_t + A_t F(k_t) + k_t - c_t - B_{t+1} - k_{t+1}] \right\}$$

And her decision problem is characterized by three sets of first order conditions, one each for consumption, bonds, and capital accumulation

$$\begin{aligned}
\frac{\partial L}{\partial c_t} &= 0 \quad \Leftrightarrow \quad \beta^t u'(c_t) = \lambda_t \\
\frac{\partial L}{\partial B_{t+1}} &= 0 \quad \Leftrightarrow \quad \lambda_t = E_t\{(1+r)\lambda_{t+1}\} \\
\frac{\partial L}{\partial k_{t+1}} &= 0 \quad \Leftrightarrow \quad \lambda_t = E_t\{[1 + A_{t+1}F'(k_{t+1})]\lambda_{t+1}\}
\end{aligned}$$

The last two of these conditions are asset pricing equations. In each case, the consumer foregoes some consumption and pays a cost λ_t. At an optimum, this price must be equalized with a benefit. Indeed, the benefits from acquiring bonds or holding capital must be the same. In each case, part of the benefit comes from λ_{t+1}. In the case of bonds, there is a sure return $1+r$ per λ_{t+1}. In the case of capital, there is a risky return $1 + A_{t+1}F'(k_{t+1})$ that depends on the marginal product of capital at $t+1$ which is unknown as of date t.

Rewriting the first order conditions gives

$$\begin{aligned}
u'(c_t) &= \beta E_t\{(1+r)u'(c_{t+1})\} \\
u'(c_t) &= \beta E_t\{[1 + A_{t+1}F'(k_{t+1})]u'(c_{t+1})\}
\end{aligned}$$

which imply

$$\frac{1}{1+r} = E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\}$$

and

$$\begin{aligned}
1 &= E_t \left\{ [1 + A_{t+1}F'(k_{t+1})] \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\} \\
&= E_t \{1 + A_{t+1}F'(k_{t+1})\} \, E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\} + \mathrm{Cov}_t \left[\{A_{t+1}F'(k_{t+1})\}, \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\} \right]
\end{aligned}$$

(Using the definition of a covariance, $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$ and that $\text{Cov}(z + X, Y) = \text{Cov}(X,Y)$ for any constant z). Now using the price of a bond, $(1+r)^{-1}$ to simplify this last expression

$$1 = E_t \left\{1 + A_{t+1}F'(k_{t+1})\right\} \frac{1}{1+r} + \text{Cov}_t \left[\{A_{t+1}F'(k_{t+1})\}, \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\} \right]$$

or

$$E_t \left\{A_{t+1}F'(k_{t+1})\right\} = r - \beta(1+r)\text{Cov}_t \left[\{A_{t+1}F'(k_{t+1})\}, \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \right\} \right]$$

(which uses $\text{Cov}(zX,Y) = z\text{Cov}(X,Y)$ for any constant z).

Thus capital accumulation takes place until the expected marginal product of capital is set equal to the riskless rate minus a conditional covariance term. Since a positive technology shock will increase consumption and lower the marginal utility of consumption, this covariance term is negative. The expected marginal product of capital is the riskless rate plus a risk premium

$$E_t \left\{ r_{t+1}^{\text{risky}} \right\} = r^{\text{riskless}} + \text{risk premium}_t$$

A high risk premium is demanded of an asset that is poor from an insurance perspective, i.e., an asset that has a high return only when the marginal utility of consumption is low (and consumption itself is relatively high). In general, this risk premium is time-varying.

Chris Edmond, 18 August 2003