316-632 International Monetary Economics — Note 1

Let me briefly recap on the dynamics of current accounts in small open economies. Consider the problem of a representative consumer in a country that is perfectly integrated with world capital markets and that takes as given a constant world real interest rate $r > 0$. The consumer is born at date $t = 0$ and lives until $t = T$ with preferences $U(c)$ over the consumption vector

$$c = (c_0, c_1, \cdots, c_T)$$

For simplicity, I assume that the consumer discounts the future at a geometric rate and has time-separable preferences of the form

$$\begin{aligned} U(c) &= u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots + \beta^T u(c_T) \\ &= \sum_{t=0}^{T} \beta^t u(c_t) \end{aligned}$$

This consumer faces a sequence of *flow budget constraints*, each of the form

$$B_{t+1} - B_t = rB_t + y_t - c_t - i_t - g_t$$

The change in net foreign assets $B_{t+1} - B_t$ is the country's current account *balance*. If $B_{t+1} > B_t$, the country runs a current account surplus in date t while if $B_{t+1} < B_t$, the country runs a current account deficit. Government expenditure $\{g_t\}$ is a known exogenous sequence. The sum $rB_t + y_t$ is Gross National Product (GNP) with Gross Domestic Product (GDP) denoted by y_t. GDP is determined by the physical capital stock (labor is not a factor of production) according to a production function

$$y_t = F(k_t)$$

Investment is the change in the capital stock net of depreciation, $i_t = k_{t+1} - k_t - \delta k_t$ where δ denotes the depreciation rate. I will assume that $\delta = 0$ so that physical capital never depreciates. This implies that

$$i_t = k_{t+1} - k_t$$

The initial capital stock $k_0 > 0$ is a given parameter of the model. Choosing an investment plan is equivalent to choosing a sequence of capital installations $\{k_{t+1}\}$.

Intertemporal budget constraint

The sequence of flow budget constraints can be integrated to give a single *intertemporal* (or *present value*) *budget constraint*. This is done by recursive substitution. The basic idea is to continuously eliminate the future asset terms, B_{t+1}, from the constraints. Mechanically,

$$\begin{aligned} B_1 &= (1+r)B_0 + y_0 - c_0 - i_0 - g_0 \\ B_2 &= (1+r)B_1 + y_1 - c_1 - i_1 - g_1 \end{aligned}$$

Substituting B_1 into the second equation gives

$$B_2 = (1+r)[(1+r)B_0 + y_0 - c_0 - i_0 - g_0] + y_1 - c_1 - i_1 - g_1$$

Now write out an expression for B_3

$$\begin{aligned} B_3 &= (1+r)B_2 + y_2 - c_2 - i_2 - g_2 \\ &= (1+r)\{(1+r)[(1+r)B_0 + y_0 - c_0 - i_0 - g_0] + y_1 - c_1 - i_1 - g_1\} + y_2 - c_2 - i_2 - g_2 \end{aligned}$$

More generally, for any t

$$B_{t+1} = (1+r)^{t+1} B_0 + \sum_{s=0}^{t} (1+r)^{t-s} (y_s - c_s - i_s - g_s)$$

Dividing throughout by the common factor $(1+r)^t$, evaluating at $t = T$ and rearranging gives the intertemporal budget constraint

$$\sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t c_t + \left(\frac{1}{1+r}\right)^T B_{T+1} = (1+r)B_0 + \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t (y_t - i_t - g_t)$$

Intertemporal optimization

The consumer's problem is to choose a consumption vector c and an investment plan to maximize her utility function subject to the budget constraint, the production function, and the definition of investment. The Lagrangian for this problem is

$$\begin{aligned} L &= \sum_{t=0}^{T} \beta^t u(c_t) + \lambda \left[(1+r)B_0 + \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t (y_t - i_t - g_t - c_t) - \left(\frac{1}{1+r}\right)^T B_{T+1} \right] \\ &= \sum_{t=0}^{T} \beta^t u(c_t) + \lambda \left[(1+r)B_0 + \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t (F(k_t) - (k_{t+1} - k_t) - g_t - c_t) - \left(\frac{1}{1+r}\right)^T B_{T+1} \right] \end{aligned}$$

where λ denotes a Lagrange multiplier. The first order conditions that characterize this problem include

$$\begin{aligned} \frac{\partial L}{\partial c_t} &= 0 \quad \Leftrightarrow \quad \beta^t u'(c_t) - \lambda \left(\frac{1}{1+r}\right)^t = 0 \qquad \text{each } t \\ \frac{\partial L}{\partial k_{t+1}} &= 0 \quad \Leftrightarrow \quad -\lambda \left(\frac{1}{1+r}\right)^t + \lambda \left(\frac{1}{1+r}\right)^{t+1} F'(k_{t+1}) = 0 \qquad \text{each } t \end{aligned}$$

(We can also derive the obvious conclusion that $B_{T+1} = 0$ by noting that there is a cost to acquiring assets in the last period but no offsetting benefit). The optimality conditions can be rearranged to give the familiar consumption-smoothing condition and the requirement that investment take place up to the point where the marginal product of capital equals the given world real interest rate. In this notation,

$$\begin{aligned} u'(c_t) &= \beta(1+r)u'(c_{t+1}) \\ r &= F'(k_{t+1}) \end{aligned}$$

We can invert the last condition to solve for the capital stock in terms of r. When r is constant, k_{t+1} is constant at some $k = (F')^{-1}(r)$ too. With a constant exogenous world real interest rate, capital accumulation is not determined simultaneously with consumption.

The consumption function

To solve for consumption, we have to combine the first order condition

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

with the budget constraint

$$\sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t c_t = (1+r)B_0 + \sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t (y_t - i_t - g_t)$$

(I have used the fact that $B_{T+1} = 0$).

Example 1.

Suppose that $\beta(1+r) = 1$ so that the discount rate $\rho \equiv \beta^{-1} - 1$ is equal to the world real interest rate r. Then

$$u'(c_t) = u'(c_{t+1})$$

implies that

$$c_t = c_{t+1} = \bar{c} \qquad \text{each } t$$

We still need to solve for this level $\bar{c}$ of consumption. Substituting into the budget constraint

$$\sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t \bar{c} = (1+r)B_0 + \sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t (y_t - i_t - g_t)$$

Since $\bar{c}$ is the same for all t we can pull it outside of the sum

$$\bar{c}\sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t = (1+r)B_0 + \sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t (y_t - i_t - g_t)$$

Now evaluating the sum on the left hand side gives (from a standard formula for geometric series, $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$ for $0 < x < 1$),

$$\sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t = \frac{1-\left(\frac{1}{1+r}\right)^{T+1}}{1-\left(\frac{1}{1+r}\right)} = \left[1-(1+r)^{-(T+1)}\right]\left(\frac{1+r}{r}\right)$$

So our consumption function is

$$\bar{c} = \frac{1}{1-(1+r)^{-(T+1)}}\frac{r}{1+r}\left[(1+r)B_0 + \sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t (y_t - i_t - g_t)\right]$$

(Recall that $k = (F')^{-1}(r)$ so that everything on the right hand side can be written in terms of exogenous variables). This is a version of the permanent income hypothesis. The main determinant of consumption is intertemporal wealth (or "permanent income"). The marginal propensity to consume out of wealth depends on T As T become large,

$$\begin{aligned}
\bar{c} &= \lim_{T\to\infty}\left\{\frac{1}{1-(1+r)^{-(T+1)}}\frac{r}{1+r}\left[(1+r)B_0+\sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t(y_t-i_t-g_t)\right]\right\} \\
&= \frac{r}{1+r}\left[(1+r)B_0+\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t(y_t-i_t-g_t)\right]
\end{aligned}$$

Thus in the long-horizon limit, consumption is simply proportional to intertemporal wealth.

The infinite-horizon model

In this case, the consumer's preferences are ordered by

$$\begin{aligned}
U(c) &= u(c_0)+\beta u(c_1)+\beta^2 u(c_2)+\cdots \\
&= \sum_{t=0}^{\infty}\beta^t u(c_t)
\end{aligned}$$

which is well defined if $0<\beta<1$ and the period utility function is either i) bounded, or ii) such that consumption does not grow too fast. The natural infinite-horizon budget constraint is

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t c_t+\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T B_{T+1}=(1+r)B_0+\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t(y_t-i_t-g_t)$$

In order to make this well defined, it is standard practice to impose a "no-Ponzi-game" constraint of the form

$$\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T B_{T+1}\geq 0$$

to ensure that the consumer cannot roll-over debt continuously. This leads to the requirement that the present value of consumption satisfy

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t c_t\leq(1+r)B_0+\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t(y_t-i_t-g_t)$$

(Of course, if $u(c_t)$ is strictly increasing in c_t this will always hold with equality). The same first order conditions can be obtained, namely,

$$\begin{aligned}
u'(c_t) &= \beta(1+r)u'(c_{t+1}) \\
r &= F'(k_{t+1})
\end{aligned}$$

Example 2.

Now suppose that period utility has the isoelastic form

$$u(c) = \frac{c^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}}$$

where $\sigma > 0$ denotes the constant intertemporal elasticity of substitution of the consumer. Then the marginal utility of consumption at date t is $u'(c_t) = c_t^{-\frac{1}{\sigma}}$ so that the consumption smoothing condition can be written

$$c_t^{-\frac{1}{\sigma}} = \beta(1+r)c_{t+1}^{-\frac{1}{\sigma}}$$

or

$$c_{t+1} = \beta^{\sigma}(1+r)^{\sigma}c_t$$

If $\beta(1+r) = 1$ we again have that $c_{t+1} = c_t$. More generally, we have

$$c_t = \left[\beta^{\sigma}(1+r)^{\sigma}\right]^t c_0$$

so that consumption at any date is a scaled up or down version of consumption at date zero. As before, if the consumer is relatively patient — so that she discounts less than the world interest rate — she has a growing consumption path, while if the consumer is relatively impatient she has a shrinking consumption path.

Now combine the formula $c_t = \left[\beta^{\sigma}(1+r)^{\sigma}\right]^t c_0$ with the intertemporal budget constraint

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t c_t = (1+r)B_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t (y_t - i_t - g_t)$$

to determine the initial consumption c_0. Obviously,

$$c_0\sum_{t=0}^{\infty}\left[\beta^{\sigma}(1+r)^{\sigma-1}\right]^t = (1+r)B_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t (y_t - i_t - g_t)$$

But

$$\sum_{t=0}^{\infty}\left[\beta^{\sigma}(1+r)^{\sigma-1}\right]^t = \frac{1}{1-\beta^{\sigma}(1+r)^{\sigma-1}} = \frac{1+r}{1+r-\beta^{\sigma}(1+r)^{\sigma}}$$

(Assuming that $0 < \beta^{\sigma}(1+r)^{\sigma-1} < 1$). Hence

$$c_0 = \frac{r+v}{1+r}\left[(1+r)B_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t (y_t - i_t - g_t)\right]$$

where the number v is

$$v \equiv 1 - \beta^{\sigma}(1+r)^{\sigma}$$

v summarizes the influence of σ and of $\beta(1+r) \neq 1$. If $\beta(1+r) = 1$, we have the same consumption function as in Example 1 with $v = 0$.

Dynamics of the current account

Let me introduce some notation which is helpful for discussing present value budget constraints. For any variable x, let $\tilde{x}_t$ denote the *permanent value* of x at date t. This is the solution to

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}\tilde{x}_t=\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}x_s$$

For a given world real interest rate $r>0$, this is a mapping from the sequence $\{x_s\}$ to the single number $\tilde{x}_t$. Specifically,

$$\tilde{x}_t=\frac{r}{1+r}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}x_s$$

(Using $\sum_{i=0}^{\infty}z^i=(1-z)^{-1}$ for $0<z<1$ and rearranging). Hence the permanent value is a measure of the central tendency of the sequence $\{x_s\}$ weighted by the discount factors.

Now suppose that $\beta(1+r)=1$ as in Example 1. Then as in that example, the consumption function is

$$c_0=\frac{r}{1+r}\left[(1+r)B_0+\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}(y_t-i_t-g_t)\right]$$

Or at <u>any</u> initial date t,

$$c_t=\frac{r}{1+r}\left[(1+r)B_t+\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(y_s-i_s-g_s)\right]$$

(This follows using the changes of variable $0\mapsto t$ and $t\mapsto s-t$). In terms of permanent values, this is just

$$c_t=rB_t+\tilde{y}_t-\tilde{\imath}_t-\tilde{g}_t$$

Now recall the flow budget constraint

$$B_{t+1}-B_t=rB_t+y_t-c_t-i_t-g_t$$

and eliminate consumption using $c_t=rB_t+\tilde{y}_t-\tilde{\imath}_t-\tilde{g}_t$. This gives

$$B_{t+1}-B_t=(y_t-\tilde{y}_t)-(i_t-\tilde{\imath}_t)-(g_t-\tilde{g}_t)$$

In this example, the current account $B_{t+1}-B_t$ is the sum of three terms, each the difference between a variable and its permanent value. If y_t is relatively high, so that $y_t>\tilde{y}_t$, there will (*ceteris paribus*) be a current account surplus, $B_{t+1}>B_t$. Similarly, if i_t or g_t is relatively, high, there will be a current account deficit. Over time, of course, the present value of current accounts must be zero.

Chris Edmond, 11 August 2003